Losing Control of your Network? Try Resilience Theory

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Abstract

Resilience of cyber-physical networks to unexpected failures is a critical need widely recognized across domains. For instance, power grids, telecommunication networks, transportation infrastructures and water treatment systems have all been subject to disruptive malfunctions and catastrophic cyber-attacks. Following such adverse events, we investigate scenarios where a network node suffers a loss of control authority over some of its actuators. These actuators are not following the controller's commands and are instead producing undesirable outputs. The repercussions of such a loss of control can propagate and destabilize the whole network despite the malfunction occurring at a single node. To assess system vulnerability, we establish resilience conditions for networks with a subsystem enduring a loss of control authority over some of its actuators. Furthermore, we quantify the destabilizing impact on the overall network when such a malfunction perturbs a nonresilient subsystem. We illustrate our resilience conditions on two academic examples and on the classical IEEE 39-bus system.

1 Introduction

Resilience of cyber-physical networks to catastrophic events is a crucial challenge, widely recognized across government levels [1, 2] and research fields [3, 4]. Natural disasters, terrorist acts, and cyber-attacks all have the potential to paralyze the cyber-physical infrastructures upon which our society inconspicuously relies, such as power grids, telecommunication networks, sewage systems and transportation infrastructures [5, 6, 7]. Motivated by these issues, we investigate the resilience of linear networks to partial loss of control authority over their actuators. This class of malfunction, initially introduced in [8], is characterized by some of the actuators producing uncontrolled and thus possibly undesirable inputs with their full capabilities [9]. This framework encompasses scenarios where actuators are taken over, for instance, by a cyber-attack [5, 6, 7], and scenarios where the actuators become unresponsive or damaged, for instance, by a software bug [10].

Building on fault-detection and isolation theory [11] coupled with cyber-attack detection [5] and state reconstruction methods [6], we assume that the controller has real-time readings of the outputs of the malfunctioning actuators. Our objective is then to assess the network's resilient stabilizability in the face of these possibly undesirable inputs [9].

Contrary to previous works [9, 12, 3], we consider actuators with bounded amplitude instead of $\mathcal{L}_2$ constraints for applicability purposes. This choice also prevents the direct use of work [6] which studies the observability properties of cyber-physical systems under unbounded adversarial attacks.

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Previous works on resilience theory [13, 14] quantified the degradation of the reachability capabilities of an isolated system enduring a partial loss of control authority over its actuators. When such a malfunctioning system is not isolated, but belongs instead to a network of interconnected systems, the loss of control can start a chain reaction capable of destabilizing the entire network. The main contribution of this work is to study these destabilizing repercussions. Albeit using a different setting, works [3, 15] also study the resilience of networks. Relying on observability and controllability, these works quantify the network's capabilities to detect a perturbed state and steer it back to its nominal value [3, 15]. Because the approach of such papers does not model the perturbation it cannot handle a malfunctioning actuator producing undesirable inputs. Additionally, works [3, 15] require $\mathcal{L}_2$ inputs, which is incompatible with numerous applications like a power grid where voltage and intensities must remain in a specified range.

The contributions of this work are threefold. First, we establish resilient stabilizability conditions for networks with a subsystem suffering a loss of control authority over some of its actuators. Second, we quantify the maximal magnitude of undesirable inputs that can be applied to a nonresilient subsystem without destabilizing the rest of the fully-actuated network. Finally, we extend this quantification to underactuated networks by relying on a feedback controller.

The remainder of this paper is organized as follows. Section 2 introduces the network dynamics and states our problems of interest. Building on prior resilience work [14], Section 3 establishes stabilizability conditions for resilient linear networks. Section 4 quantifies the resilient stabilizability of networks where a loss of control authority impacts a nonresilient subsystem. We illustrate our work on two academic examples and on the linearized IEEE 39-bus system in Section 5. Finally, Appendix A gathers supporting lemmata.

Notation: We denote the integer interval from a to b, inclusive, with $[\![a, b]\!]$. For a set $\Lambda \subseteq \mathbb{C}$, we say that $Re(\Lambda) \leq 0$ (resp. $Re(\Lambda) = 0$) if the real part of each $\lambda \in \Lambda$ verifies $Re(\lambda) \leq 0$ (resp. $Re(\lambda) = 0$). The norm of a matrix A is $\|A\| := \sup\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max\limits_{\|x\|=1} \|Ax\|$ and the set of its eigenvalues is $\Lambda(A)$. If A is positive definite, denoted $A \succ 0$, then its extremal eigenvalues are λ_{min}^A and λ_{max}^A and A generates a vector norm $\|x\|_A := \sqrt{x^\top A x}$. The controllability matrix of pair (A, B) is $\mathcal{C}(A, B) := \big[B \, AB \, \dots \, A^{n-1}B\big]$. For a matrix $B \in \mathbb{R}^{n \times m}$ and a set $\mathcal{U} \subseteq \mathbb{R}^m$ we use $B\mathcal{U}$ to denote the set $B\mathcal{U} := \big\{Bu : u \in \mathcal{U}\big\} \subseteq \mathbb{R}^n$. The block diagonal matrix composed of matrices $A_1, \dots, A_n$ is denoted by $\mathrm{diag}(A_1, \dots, A_n)$. The zero matrix of size $n \times m$ is denoted by $0_{n,m}$, the identity matrix of size n is I_n, and the vector of ones is $\mathbf{1}_n \in \mathbb{R}^n$. The convex hull of a set $\mathcal{Z}$ is denoted by $\mathrm{co}(\mathcal{Z})$, its interior by $\mathrm{int}(\mathcal{Z})$, and its orthogonal complement by $\mathcal{Z}^\perp$. The set of time functions taking value in $\mathcal{Z}$ is denoted $\mathcal{F}(\mathcal{Z}) := \big\{f : [0, +\infty) \to \mathcal{Z}\big\}$. The Minkowski addition of sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ is $\mathcal{X} \oplus \mathcal{Y} := \big\{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\big\}$ and their Minkowski difference is $\mathcal{X} \ominus \mathcal{Y} := \big\{z \in \mathbb{R}^n : \{z\} \oplus \mathcal{Y} \subseteq \mathcal{X}\big\}$. The operator $\mathrm{span}(\cdot)$ maps a set of vectors to their linear span.

2 Networks preliminaries

In this section we introduce the network under study and our two problems of interest. Inspired by [16], we consider a network of N linear subsystems of dynamics

$$\dot{x}_1(t) = A_1 x_1(t) + \bar{B}_1 \bar{u}_1(t) + \sum_{k\,\in\,\mathcal{N}_1} L_{1,k} y_k(t), \qquad y_1(t) = F_1 x_1(t), \qquad x_1(0) = x_1^0 \in \mathbb{R}^{n_1}, \tag{1-1}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots$$

$$\dot{x}_N(t) = A_N x_N(t) + \bar{B}_N \bar{u}_N(t) + \sum_{k\,\in\,\mathcal{N}_N} L_{N,k} y_k(t), \quad y_N(t) = F_N x_N(t), \quad x_N(0) = x_N^0 \in \mathbb{R}^{n_N}, \tag{1-N}$$

where $x_i \in \mathbb{R}^{n_i}$, $\bar{u}_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{q_i}$ are respectively the state, the control input and the output of subsystem $i \in [\![1, N]\!]$. Additionally, $\mathcal{N}_i \subseteq [\![1, N]\!]$ is the set of neighbors of subsystem i with $i \notin \mathcal{N}_i$, while $A_i \in \mathbb{R}^{n_i \times n_i}$, $\bar{B}_i \in \mathbb{R}^{n_i \times m_i}$, $L_{i,k} \in \mathbb{R}^{n_i \times q_i}$ and $F_i \in \mathbb{R}^{q_i \times n_i}$ are constant matrices. The set of admissible control inputs for subsystem i is the unit hypercube of $\mathbb{R}^{m_i}$, i.e., $\bar{u}_i(t) \in \bar{\mathcal{U}}_i := [-1, 1]^{m_i}$. To alleviate the notations, we introduce matrices $D_{i,k} := L_{i,k} F_k$ to represent the connection between subsystems i and k for $i \in [\![1, N]\!]$ and $k \in \mathcal{N}_i$. Then,

$$\dot{x}_1(t) = A_1 x_1(t) + \bar{B}_1 \bar{u}_1(t) + \sum_{k\,\in\,\mathcal{N}_1} D_{1,k} x_k(t), \qquad x_1(0) = x_1^0 \in \mathbb{R}^{n_1}, \tag{2-1}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\dot{x}_N(t) = A_N x_N(t) + \bar{B}_N \bar{u}_N(t) + \sum_{k\,\in\,\mathcal{N}_N} D_{N,k} x_k(t), \qquad x_N(0) = x_N^0 \in \mathbb{R}^{n_N}. \tag{2-N}$$

Let us now define our notion of finite-time *component stabilizability.*

Definition 1. *Tuple* $(A, \bar{B}, \bar{\mathcal{U}})$ *is* stabilizable *(resp.* controllable*) if there exists a time $T \geq 0$ and an admissible control signal $\bar{u} \in \mathcal{F}(\bar{\mathcal{U}})$ driving the state of system $\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t)$ from any $x^0 \in \mathbb{R}^n$ to $x(T) = 0$ (resp. to any $x_{goal} \in \mathbb{R}^n$).*

Following Definition 1, the stabilizability and controllability of tuple $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ characterize subsystem (2-i) as if it was isolated from its neighbors. These are local properties from which we will want to derive the associated overall network properties. To do so, we define the network state X and control input $\bar{u}$ as

$$X := \big(x_1, x_2, \ldots, x_N\big) \in \mathbb{R}^{n_\Sigma} \quad \text{and} \quad \bar{u}(t) := \big(\bar{u}_1(t), \ldots, \bar{u}_N(t)\big) \in \bar{\mathcal{U}} := \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_N \subseteq \mathbb{R}^{m_\Sigma}$$

with $n_\Sigma := n_1 + \ldots + n_N$ and $m_\Sigma := m_1 + \ldots + m_N$. Network dynamics (2) can then be written more concisely as

$$\dot{X}(t) = (A + D)X(t) + \bar{B}\bar{u}(t), \qquad X(0) = X_0 = \big(x_1^0, \ldots, x_N^0\big) \in \mathbb{R}^{n_\Sigma}, \tag{3}$$

with the constant matrices $A := \operatorname{diag}(A_1, \ldots, A_N)$, $\bar{B} := \operatorname{diag}\big(\bar{B}_1, \ldots, \bar{B}_N\big)$ and $D := \big(D_{i,j}\big)_{(i,j)\in[\![1,N]\!]}$ with $D_{i,k} = 0$ if $k \notin \mathcal{N}_i$. Since our objective is to investigate connected networks, we assume that $D \neq 0$.

Following, for instance, a software bug or an adversarial attack [6, 15], assume that subsystem (2-N) suffers a loss of control authority over a number $p_N \in [\![1, m_N]\!]$ of its m_N actuators. We then split the nominal input $\bar{u}_N$ between the remaining controlled inputs $u_N \in \mathcal{F}(\mathcal{U}_N)$, $\mathcal{U}_N = [-1, 1]^{m_N - p_N}$ and the uncontrolled and possibly undesirable inputs $w_N \in \mathcal{F}(\mathcal{W}_N)$, $\mathcal{W}_N = [-1, 1]^{p_N}$. We split accordingly matrix $\bar{B}_N$ into $B_N \in \mathbb{R}^{n_N \times (m_N - p_N)}$ and $C_N \in \mathbb{R}^{n_N \times p_N}$, so that the dynamics of subsystem (2-N) become

$$\dot{x}_N(t) = A_N x_N(t) + B_N u_N(t) + C_N w_N(t) + \sum_{k \in \mathcal{N}_N} D_{N,k} x_k(t), \qquad x_N(0) = x_N^0 \in \mathbb{R}^{n_N}. \tag{4}$$

We adopt the resilience framework of [12, 17] where controller $u_N(t)$ has real-time knowledge of the undesirable inputs $w_N(t)$ thanks to sensors located on each actuator. This assumption of real-time knowledge was relaxed in [18] by considering a controller inflicted by a constant actuation delay. Beyond this additional layer of complexity, the resilience conditions were extremely similar to those with immediate knowledge of the perturbations, which is why we make this simplifying assumption.

Our central objective is to study how the partial loss of control authority over actuators of subsystem (2-N) affects the *stabilizability* and the *controllability* of the whole network. To adapt these properties to malfunctioning system (4), we first need the notion of *resilient reachability* introduced in [9].

Definition 2. *A target $x_{goal} \in \mathbb{R}^n$ is* resiliently reachable *from $x(0) \in \mathbb{R}^n$ by malfunctioning system $\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$ if for all $w \in \mathcal{F}(\mathcal{W})$, there exists $T \geq 0$ and $u \in \mathcal{F}(\mathcal{U})$ such that $u(t)$ only depends on $w([0,t])$ and the solution exists, is unique and $x(T) = x_{goal}$.*

Definition 3. *Tuple $(A, B, C, \mathcal{U}, \mathcal{W})$ is* resiliently stabilizable *(resp.* resilient*) if $0 \in \mathbb{R}^n$ (resp. every $x_{goal} \in \mathbb{R}^n$) is resiliently reachable from any $x(0) \in \mathbb{R}^n$ by malfunctioning system $\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$.*

Network dynamics (3) are also impacted by the loss of control authority in subsystem (2-N). We define the network control input $u(t) := \big(\bar{u}_1(t), \ldots, \bar{u}_{N-1}(t), u_N(t)\big) \in \mathcal{U} := \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_{N-1} \times \mathcal{U}_N \subseteq \mathbb{R}^{m_\Sigma - p_N}$. Network dynamics (3) then become

$$\dot{X}(t) = (A + D)X(t) + Bu(t) + Cw_N(t), \qquad X(0) = X_0 = \big(x_1^0, \ldots, x_N^0\big) \in \mathbb{R}^{n_\Sigma}, \tag{5}$$

with the constant matrices $B = \mathrm{diag}\big(\bar{B}_1, \ldots, \bar{B}_{N-1}, B_N\big)$ and $C = \begin{pmatrix} 0_{(n_\Sigma - n_N) \times p_N} \\ C_N \end{pmatrix}$.

Definition 4. *Network* (5) *is* resiliently stabilizable *(resp.* resilient*) if tuple $\big(A + D, B, C, \mathcal{U}, \mathcal{W}_N\big)$ is resiliently stabilizable (resp. resilient).*

We are now led to the following problems of interest.

Problem 1. *Assuming that $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is resiliently stabilizable and $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ is stabilizable for $i \in [\![1, N-1]\!]$, under what conditions is network* (5) *resiliently stabilizable?*

Problem 2. *Assuming that $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is resilient and $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ is controllable for $i \in [\![1, N-1]\!]$, under what conditions is network* (5) *resilient?*

After investigating the ideal cases of Problems 1 and 2 where tuple $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is resilient, we will consider the more problematic scenario where it is not resilient and study whether the other subsystems of the network are stabilizable or controllable despite the perturbations arising from the coupling with malfunctioning subsystem (4). Let $\chi(t)$ denote the combined state of all other subsystems, i.e., $\chi(t) := \big(x_1(t), \ldots, x_{N-1}(t)\big)$ Then,

$$\dot{\chi}(t) = \hat{A}\chi(t) + \hat{B}\hat{u}(t) + \hat{D}\chi(t) + D_{-,N}x_N(t), \tag{6}$$

with $\chi_0 := \big(x_1^0, \ldots, x_{N-1}^0\big)$, $\hat{A} := \mathrm{diag}\big(A_1, \ldots, A_{N-1}\big)$, $\hat{B} := \mathrm{diag}\big(\bar{B}_1, \ldots, \bar{B}_{N-1}\big)$, and $\hat{u}(t) :=$

$\big(\bar{u}_1(t), \dots, \bar{u}_{N-1}(t)\big) \in \hat{\mathcal{U}} := \bar{\mathcal{U}}_1 \times \dots \times \bar{\mathcal{U}}_{N-1} = [-1,1]^{m_\Sigma - m_N}$. We also split matrix D accordingly:

$$D = \left(\begin{array}{cccccc|c} 0 & D_{1,2} & \dots & D_{1,N\text{-}2} & D_{1,N\text{-}1} & D_{1,N} \\ D_{2,1} & 0 & & & D_{2,N\text{-}1} & D_{2,N} \\ \vdots & & \ddots & & \vdots & \vdots \\ D_{N\text{-}2,1} & & & 0 & D_{N\text{-}2,N\text{-}1} & D_{N\text{-}2,N} \\ D_{N\text{-}1,1} & D_{N\text{-}1,2} & \dots & D_{N\text{-}1,N\text{-}2} & 0 & D_{N\text{-}1,N} \\ \hline D_{N,1} & \dots & & \dots & D_{N,N\text{-}1} & 0 \end{array}\right) := \left(\begin{array}{c|c} \hat{D} & D_{-,N} \\ \hline D_{N,-} & 0 \end{array}\right). \quad (7)$$

Then, the last row of D without the last diagonal block is $D_{N,-}$, while the last column of D without the last diagonal block is $D_{-,N}$.

Definition 5. *System* (6) *is* resiliently stabilizable *(resp.* resilient*) if for every $X_0 \in \mathbb{R}^{n_\Sigma}$ (resp. and every $\chi_{goal} \in \mathbb{R}^{n_\Sigma - n_N}$) and every $w_N \in \mathcal{F}(\mathcal{W}_N)$ there exists $T \geq 0$ and $u \in \mathcal{F}(\mathcal{U})$ such that the solution to* (5) *exists, is unique and $\chi(T) = 0$ (resp. $\chi(T) = \chi_{goal}$).*

Definition 5 considers the joint stabilizability of subsystems 1 to $N-1$ despite any undesirable input w_N perturbing their combined state χ through the coupling term $D_{-,N}x_N$ in (6). The resilient stabilizability of subsystem (6) depends on the initial state X_0 of network (5) and on the undesirable input w_N perturbing state χ through the coupling term $D_{-,N}x_N$ in (6). We can then state our third problem of interest.

Problem 3. *Assuming that $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is* not *resiliently stabilizable and $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ is stabilizable for $i \in [\![1, N-1]\!]$, under what conditions is system* (6) *resiliently stabilizable?*

Problem 4. *Assuming that $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is* not *resilient and $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ is controllable for $i \in [\![1, N-1]\!]$, under what conditions is system* (6) *resilient?*

Note that Problems 3 and 4 do not try to resiliently control subsystem (4) along with the other subsystems. Indeed, the only way to do so would rely on the coupling term $\sum D_{N,k}x_k$, which is going to 0 as the other subsystems are getting stabilized. Therefore, malfunctioning network (5) is not resiliently stabilizable when tuple $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is not resiliently stabilizable. We start by investigating Problem 2.

3 Stabilizability of resilient networks

In this section, we build on several background results from stabilizability and resilience theories to tackle Problems 1 and 2.

3.1 Background results

We consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad \bar{u}(t) \in \bar{\mathcal{U}}, \quad (8)$$

with $A \in \mathbb{R}^{n \times n}$ and $\bar{B} \in \mathbb{R}^{n \times m}$ constant matrices and $\bar{\mathcal{U}} = [-1,1]^m$. The controllability and stabilizability of system (8) can be assessed with Corollaries 3.6 and 3.7 of Brammer [19], restated here together as Theorem 1.

Theorem 1 (Brammer's conditions [19])**.** *If $\bar{\mathcal{U}} \cap \ker(\bar{B}) \neq \emptyset$ and $\mathrm{int}(\mathrm{co}(\bar{\mathcal{U}})) \neq \emptyset$, then system* (8) *is stabilizable (resp. controllable) if and only if $\mathrm{rank}\big(\mathcal{C}(A, \bar{B})\big) = n$, $Re\big(\lambda(A)\big) \leq 0$ (resp. $= 0$) and there is no real eigenvector v of $A^\top$ satisfying $v^\top \bar{B}\bar{u} \leq 0$ for all $\bar{u} \in \bar{\mathcal{U}}$.*

When $0 \in \operatorname{int}(\bar{\mathcal{U}})$, Theorem 1 boils down to Sontag's stabilizability condition [20] as the eigenvector criteria can be removed.

Theorem 2 (Sontag's condition [20]). *If $0 \in \operatorname{int}(\bar{\mathcal{U}})$, then system (8) is stabilizable (resp. controllable) if and only if $\operatorname{rank}\big(\mathcal{C}(A, \bar{B})\big) = n$ and $Re\big(\lambda(A)\big) \leq 0$ (resp. $= 0$).*

After a loss of control authority over p of the m actuators of system (8), the input signal $\bar{u}$ is split between the remaining controlled inputs $u \in \mathcal{F}(\mathcal{U})$, $\mathcal{U} = [-1, 1]^{m-p}$ and the uncontrolled and possibly undesirable inputs $w \in \mathcal{F}(\mathcal{W})$, $\mathcal{W} = [-1, 1]^p$. Matrix $\bar{B}$ is accordingly split into two constant matrices $B \in \mathbb{R}^{n \times (m-p)}$ and $C \in \mathbb{R}^{n \times p}$ so that the system dynamics become

$$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad u(t) \in \mathcal{U}, \quad w(t) \in \mathcal{W}. \tag{9}$$

Resilience conditions established in [14] use Hájek's approach [21] and hence require the following system associated to dynamics (9)

$$\dot{x}(t) = Ax(t) + z(t), \qquad x(0) = x_0 \in \mathbb{R}^n, \qquad z(t) \in \mathcal{Z}, \tag{10}$$

where $\mathcal{Z} \subseteq \mathbb{R}^n$ is the Minkowski difference between the set of admissible control inputs $B\mathcal{U} := \big\{Bu : u \in \mathcal{U}\big\}$ and the opposite of the set of undesirable inputs $C\mathcal{W} := \big\{Cw : w \in \mathcal{W}\big\}$, i.e.,

$$\begin{aligned}
\mathcal{Z} &:= \big[B\mathcal{U} \ominus (-C\mathcal{W})\big] \cap B\mathcal{U} \\
&= \big\{z \in B\mathcal{U} : \{z\} \oplus (-C\mathcal{W}) \subseteq B\mathcal{U}\big\} \\
&= \big\{z \in B\mathcal{U} : z - Cw \in B\mathcal{U} \text{ for all } w \in \mathcal{W}\big\}.
\end{aligned}$$

Informally, $\mathcal{Z}$ represents the control available after counteracting any undesirable input. The first resilience condition established in [14] is as follows.

Proposition 1. *If $\operatorname{int}(\mathcal{Z}) \neq \emptyset$, then system (9) is resiliently stabilizable (resp. resilient) if and only if $Re\big(\lambda(A)\big) \leq 0$ (resp. $= 0$).*

The main issue with Proposition 1 is the requirement that $\operatorname{int}(\mathcal{Z}) \neq \emptyset$ in $\mathbb{R}^n$, i.e., $\mathcal{Z}$ must be of dimension n, which implies that matrices B and $\bar{B}$ must be full rank. To remove this restrictive requirement, the work [14] relied on a matrix $Z \in \mathbb{R}^{n \times r}$ with $r := \dim(\mathcal{Z})$ such that $\operatorname{Im}(Z) = \operatorname{span}(\mathcal{Z})$.

Theorem 3 (Necessary and sufficient condition [14]). *System (9) is resiliently stabilizable (resp. resilient) if and only if $Re\big(\lambda(A)\big) \leq 0$ (resp. $= 0$), $\operatorname{rank}\big(\mathcal{C}(A, Z)\big) = n$ and there is no real eigenvector v of $A^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$.*

Corollary 1. *If $\dim(\mathcal{Z}) = \operatorname{rank}(B)$, then system (9) is resiliently stabilizable (resp. resilient) if and only if system (8) is stabilizable (resp. controllable).*

Notice how the resilience conditions only differ from the resilient stabilizability ones by further restricting the eigenvalues of A. Because of the similarity between these two concepts, from now on we will only focus on resilient stabilizability.

3.2 Stabilizability results

Malfunctioning network (5) can only be resiliently stabilizable if the network was stabilizable before the malfunction. We then start by investigating the finite-time stabilizability of initial network (3). Since $0 \in \operatorname{int}(\bar{\mathcal{U}})$, a direct application of Theorem 2 yields the following result.

Proposition 2. *Network (3) is stabilizable if and only if $\operatorname{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = n_\Sigma$ and $Re(\lambda(A + D)) \leq 0$.*

Since Problem 2 aims at relating the resilient stabilizability of malfunctioning network (5) to that of its subsystems, a preliminary step in this direction is to relate the stabilizability of initial network (3) to that of its subsystems unlike Proposition 2. We will then establish several sufficient conditions for stabilizability by studying the rank and eigenvalue conditions of Proposition 2.

First, note that having $\text{rank}\big(\mathcal{C}(A_i, \bar{B}_i)\big) = n_i$ for all $i \in [\![1, N]\!]$ does not necessarily imply $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = n_\Sigma$, even for matrices D with a small norm compared to that of A. We need a stronger condition on matrix D to ensure that the coupling between subsystems does not alter their stabilizability. We could use the Popov-Belevitch-Hautus controllability test [22] stating the equivalence between $\text{rank}\big(\mathcal{C}(A+D, \bar{B})\big) = n_\Sigma$ and $\text{rank}\big(A+D-sI, \bar{B}\big) = n_\Sigma$ for all $s \in \mathbb{C}$. In turn, this condition is equivalent to verifying whether $\bar{B}x \neq 0$ for all eigenvectors x of $A + D$. However, relating the eigenvectors of $A + D$ to those of A is very complicated, as detailed in Corollary 7.2.6. of [23].

Instead, we will prefer the *distance to uncontrollability* of [22] defined as

$$\mu(A, \bar{B}) := \min\big\{\|\Delta A, \Delta\bar{B}\| : (A+\Delta A, \bar{B}+\Delta\bar{B}) \text{ is uncontrollable}\big\} = \min\big\{\sigma_n\big(A-sI, \bar{B}\big) : s \in \mathbb{C}\big\}.$$

Since $\bar{B}$ is not affected by the coupling D, we do not need the perturbation $\Delta\bar{B}$ present in $\mu(A, \bar{B})$ and we define instead

$$\mu_{\bar{B}}(A) := \min\big\{\|\Delta A\| : (A + \Delta A, \bar{B}) \text{ is uncontrollable}\big\}.$$

Note that $\mu_{\bar{B}}(A) \geq \mu(A, \bar{B})$ as shown in Lemma 1.

From [24], we also introduce the *real stability radius* of A

$$r_\mathbb{R}(A) := \inf\big\{\|D\| : D \in \mathbb{R}^{n\times n} \text{ and } A + D \text{ is unstable}\big\}.$$

To approximate $r_\mathbb{R}(A)$ numerous lower bounds are provided in [24]. We will now derive sufficient stabilizability conditions for network (3) with the following statements.

Proposition 3. *(a) If* $\text{rank}(\bar{B}_i) = n_i$ *for all* $i \in [\![1, N]\!]$*, then* $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = n_\Sigma$ *for all* $D \in \mathbb{R}^{n\times n}$*.*

(b) If there exists a matrix $F \in \mathbb{R}^{m_\Sigma \times n_\Sigma}$ *such that* $D = \bar{B}F$ *and pairs* $(A_i, \bar{B}_i)$ *are controllable, then* $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = n_\Sigma$*.*

(c) If $\|D\| < \mu_{\bar{B}}(A)$*, then* $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = n_\Sigma$*.*

(d) If $\|D\| < r_\mathbb{R}(A)$*, then* $Re\big(\lambda(A + D)\big) \leq 0$*.*

Proof. (a) Assume that $\text{rank}(\bar{B}_i) = n_i$. Because $\bar{B} = \text{diag}(\bar{B}_1, \dots, \bar{B}_N)$, we have $\text{rank}(\bar{B}) = n_\Sigma$, which yields $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = \text{rank}\big(\bar{B}, (A + D)\bar{B}, \dots\big) = n_\Sigma$.

(b) If D can be written as state feedback $D = \bar{B}F$, then $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = \text{rank}\big(\mathcal{C}(A, \bar{B})\big)$ [25]. Since A and $\bar{B}$ are block diagonal matrices,

$$\text{rank}\big(\mathcal{C}(A, \bar{B})\big) = \sum_{i=1}^{N} \text{rank}\big(\mathcal{C}(A_i, \bar{B}_i)\big) = \sum_{i=1}^{N} n_i = n_\Sigma,$$

where $\text{rank}\big(\mathcal{C}(A_i, \bar{B}_i) = n_i$ comes from the controllability of pair $(A_i, \bar{B}_i)$.

(c) By definition of $\mu_{\bar{B}}(A)$, $\|D\| < \mu_{\bar{B}}(A)$ leads to the controllability of pair $(A + D, \bar{B})$, i.e., to $\text{rank}\big(\mathcal{C}(A + D, \bar{B})\big) = n_\Sigma$.

(d) By definition of the stability radius $\|D\| < r_\mathbb{R}(A)$ leads to the stability of $A+D$, i.e., $Re(\lambda(A+D)) \leq 0$.

□

Combining statements (a), (b) or (c) of Proposition 3 with its statement (d) yields three different sufficient stabilizability conditions for network (3) thanks to Proposition 2.

Note that having $\mu_{\bar{B}}(A) > 0$ and $r_{\mathbb{R}}(A) > 0$ implicitly requires the stabilizability of all the tuples $\big(A_i, \bar{B}_i, \bar{\mathcal{U}}_i\big)$. Indeed, $\mu_{\bar{B}}(A) > 0$ requires $(A, \bar{B})$ to be controllable, i.e., each tuple $(A_i, \bar{B}_i)$ must be controllable because of the diagonal structure of A and $\bar{B}$. Similarly, $r_{\mathbb{R}}(A) > 0$ requires $Re\big(\lambda(A)\big) \leq 0$, but $\lambda(A) = \lambda(A_1) \cup \ldots \cup \lambda(A_N)$ because $A = \mathrm{diag}(A_1, \ldots, A_N)$, hence $Re\big(\lambda(A_i)\big) \leq 0$. To sum up, $\mu_{\bar{B}}(A) > 0$ and $r_{\mathbb{R}}(A) > 0$ require $\mathrm{rank}(A_i, \bar{B}_i) = n_i$ and $Re\big(\lambda(A_i)\big) \leq 0$, which are exactly the stabilizability conditions of Sontag for tuple $\big(A_i, \bar{B}_i, \bar{\mathcal{U}}_i\big)$ stated in Theorem 2.

Proposition 3 provides several stabilizability conditions for network (3). We will now address Problem 1 by studying the network dynamics after enduring a partial loss of control authority.

3.3 Resilient stabilizability results

Inspired by the work completed before Proposition 1, we define the input sets of the network $B\mathcal{U} := \big\{Bu : u \in \mathcal{U}\big\}$, $C\mathcal{W} := \big\{Cw_N : w_N \in \mathcal{W}_N\big\}$ and their Minkowski difference $\mathcal{Z} := B\mathcal{U} \ominus (-C\mathcal{W}) \subseteq \mathbb{R}^{n_\Sigma}$. Similarly, we introduce the input sets of each subsystems $\bar{B}_i\bar{\mathcal{U}}_i := \big\{\bar{B}_i\bar{u}_i : \bar{u}_i \in \bar{\mathcal{U}}_i\big\}$ for $i \in [\![1, N-1]\!]$, $B_N\mathcal{U}_N := \big\{B_N u_N : u_N \in \mathcal{U}_N\big\}$, $C_N\mathcal{W}_N := \big\{C_N w_N : w_N \in \mathcal{W}_N\big\}$ and their Minkowski difference $\mathcal{Z}_N := B_N\mathcal{U}_N \ominus (-C_N\mathcal{W}_N)$. These sets are all linked together with the following result.

Proposition 4. *The resilient control set of network* (5) *is the Cartesian product of the input sets of its subsystems:* $\mathcal{Z} = \bar{B}_1\bar{\mathcal{U}}_1 \times \ldots \times \bar{B}_{N-1}\bar{\mathcal{U}}_{N-1} \times \mathcal{Z}_N$.

Proof. We prove this equality by showing both inclusions.

Take $z = (z_1, \ldots, z_N) \in \mathcal{Z}$. We want to show that $z_i \in \bar{B}_i\bar{\mathcal{U}}_i$ for $i \in [\![1, N-1]\!]$ and that $z_N \in \mathcal{Z}_N$. Let $w_N \in \mathcal{W}_N$. Since $z \in \mathcal{Z}$, there exists $u = (\bar{u}_1, \ldots, \bar{u}_{N-1}, u_N) \in \mathcal{U} = \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_{N-1} \times \mathcal{U}_N$ such that

$$z - Cw_N = Bu = \begin{pmatrix} \bar{B}_1\bar{u}_1 \\ \vdots \\ \bar{B}_{N-1}\bar{u}_{N-1} \\ B_N u_N \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{N-1} \\ z_N - C_N w_N \end{pmatrix}$$

Then, $z_i \in \bar{B}_i\bar{\mathcal{U}}_i$ for $i \in [\![1, N-1]\!]$. Additionally, for all $w_N \in \mathcal{W}_N$ we have $z_N - C_N w_N \in B_N\mathcal{U}_N$, i.e., $z_N \in \mathcal{Z}_N$.

On the other hand, let $\bar{u}_i \in \bar{\mathcal{U}}_i$ for $i \in [\![1, N-1]\!]$, $z_N \in \mathcal{Z}_N$ and define $z = \big(\bar{B}_1\bar{u}_1, \ldots, \bar{B}_{N-1}\bar{u}_{N-1}, z_N\big)$. We want to show that $z \in \mathcal{Z}$, so we take some $w_N \in \mathcal{W}_N$. Since $z_N \in \mathcal{Z}_N$, there exists $u_N \in \mathcal{U}_N$ such that $z_N - C_N w_N = B_N u_N$. Then,

$$z - Cw_N = \begin{pmatrix} \bar{B}_1\bar{u}_1 \\ \vdots \\ \bar{B}_{N-1}\bar{u}_{N-1} \\ z_N \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ C_N \end{pmatrix} w_N = \begin{pmatrix} \bar{B}_1\bar{u}_1 \\ \vdots \\ \bar{B}_{N-1}\bar{u}_{N-1} \\ B_N u_N \end{pmatrix} \in B\mathcal{U}, \quad \text{so } z \in \mathcal{Z}.$$

□

Let us now address Problem 1 by considering the case where $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is resiliently stabilizable. Using Proposition 1 we derive a sufficient condition for resilient stabilizability.

Proposition 5. *If* $\mathrm{rank}(\bar{B}_i) = n_i$ *for all* $i \in [\![1, N-1]\!]$, $\mathrm{int}(\mathcal{Z}_N) \neq \emptyset$ *and* $\|D\| < r_{\mathbb{R}}(A)$, *then network* (3) *is resiliently stabilizable.*

Proof. Since $\mathrm{rank}(\bar{B}_i) = n_i$, $\mathrm{int}(\bar{B}_i\bar{\mathcal{U}}_i) \neq \emptyset$, so that according to Proposition 4 we have $\mathrm{int}(\mathcal{Z}) \neq \emptyset$.

By assumption, we have $\|D\| < r_{\mathbb{R}}(A)$, i.e., $Re(\lambda(A+D)) \leq 0$. Then, Proposition 1 states that network (3) is resiliently stabilizable. $\square$

Proposition 5 provides a straightforward resilient stabilizability condition for the network in a case that is similar to Proposition 3 (a). As mentioned after Proposition 1, the condition $\mathrm{int}(\mathcal{Z}_N) \neq \emptyset$ requires $\mathrm{rank}(B_N) = \mathrm{rank}(\bar{B}_N) = n_N$. Then, Proposition 5 requires all $\bar{B}_i$ to be full rank, which is very restrictive and not necessary for stabilizability. Instead, we want to use Theorem 3 to derive a less restrictive resilient stabilizability condition for the network. To use this theorem, we must first build a matrix $Z \in \mathbb{R}^{n_\Sigma \times r_\Sigma}$ with $r_\Sigma = \dim(\mathcal{Z})$ and satisfying $\mathrm{Im}(Z) = \mathrm{span}(\mathcal{Z})$. In practice, matrix Z is built by collating r_Σ linearly independent vectors from set $\mathcal{Z}$.

Proposition 6. *If $\|D\| < \min\big\{r_{\mathbb{R}}(A), \mu_Z(A)\big\}$ and there is no real eigenvector v of $(A+D)^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$, then network* (5) *is resiliently stabilizable.*

Proof. We apply Theorem 3 to network (5) and obtain that it is resiliently stabilizable if and only if $Re\big(\lambda(A+D)\big) \leq 0$, $\mathrm{rank}\,\big(\mathcal{C}(A+D,Z)\big) = n_\Sigma$ and there is no real eigenvector v of $(A+D)^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$. The eigenvalue and rank conditions are satisfied thanks to $\|D\| < \min\{r_{\mathbb{R}}(A), \mu_Z(A)\}$, while the eigenvector condition is verified by assumption. $\square$

As before, the fact that $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ are stabilizable and that $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is resiliently stabilizable, are implied by the conditions of Proposition 6.

When $\mathcal{Z}$ is not of full dimension, the eigenvector condition of Proposition 6 is difficult to verify. Indeed, the space $\mathcal{Z}^\perp$ is non-trivial and thus might encompass a real eigenvector of $A+D$ even if none of the eigenvectors of A are part of $\mathcal{Z}^\perp$. Intuitively, when D is small, the eigenvectors of $A+D$ should be 'close' to those of A. This intuition is formalized in Corollary 7.2.6. of [23], but the complexity of its statement prevents the derivation of a simple condition to be verified by A and D. Thus, we choose to remain with Propositions 5 and 6 as our solutions to Problem 1.

4 Stabilizability of nonresilient networks

In this section, we address Problem 3 by studying the network-wide repercussions resulting from the partial loss of control authority in nonresilient subsystem (4).

We now study the eventuality where $(A_N, B_N, C_N, \mathcal{U}_N, \mathcal{W}_N)$ is not resiliently stabilizable. Following Proposition 1, we consider the case where $-C_N\mathcal{W}_N \nsubseteq B_N\mathcal{U}_N$, i.e., subsystem (4) lost an actuator to which it is not resilient. Then, there are some undesirable inputs w_N that no control input u_N can overcome. Such undesirable inputs w_N can prevent stabilizability of subsystem N as demonstrated in Lemma 6 of [14].

To evaluate the resilient stabilizability of network (5), we need to study the worst-case scenario where w_N is the most destabilizing undesirable input for subsystem (4). If A_N is not Hurwitz, these destabilizing inputs w_N can drive the state x_N to infinity. In this situation, coupling terms $D_{i,N}x_N$ impacting subsystems (2-i) can become unbounded preventing to stabilize these other subsystems. We will then focus on the case where A_N is Hurwitz, so that the state x_N cannot be forced to diverge by w_N. Then, coupling term $D_{-,N}x_N$ perturbing subsystem (6) is bounded and might be counteracted if controller $\hat{B}\hat{u}$ is strong enough.

To address Problem 3, we will quantify the maximal degree of non-resilience of subsystem (4) despite which subsystem (6) remain resiliently stabilizable in the sense of Definition 5.

We start by calculating how far can w_N force state x_N despite the best u_N and the Hurwitzness of A_N.

Proposition 7. *If A_N is Hurwitz and $-C_N\mathcal{W}_N \nsubseteq B_N\mathcal{U}_N$, then for all $t \geq 0$ the following holds:*

$$\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N\tau}\beta_N(\tau)\,d\tau\right), \tag{11}$$

for all $P_N = P_N^\top \succ 0$ and $Q_N \succ 0$ such that $A_N^\top P_N + P_N A_N = -Q_N$ and with

$$\alpha_N := \frac{\lambda_{min}^{Q_N}}{2\lambda_{max}^{P_N}},\ \beta_N(t) := z_{max}^{P_N} + \|D_{N,-}\chi(t)\|_{P_N},\ z_{max}^{P_N} := \max_{w_N\in\mathcal{W}_N}\Big\{\min_{u_N\in\mathcal{U}_N}\|C_Nw_N + B_Nu_N\|_{P_N}\Big\}.$$

Proof. Since A_N is Hurwitz, there exist a symmetric $P_N \succ 0$ and $Q_N \succ 0$ such that $A_N^\top P_N + P_N A_N = -Q_N$ according to Lyapunov theory [26]. Let us consider any such pair (P_N, Q_N). Then, inspired by Example 15 of [26], we study the P_N-norm of x_N, i.e., $x_N^\top P_N x_N = \|x_N\|_{P_N}^2$ when state x_N is following dynamics (4). We obtain

$$\begin{aligned}\frac{d}{dt}\|x_N(t)\|_{P_N}^2 &= \dot{x}_N^\top P_N x_N + x_N^\top P_N \dot{x}_N \\ &= x_N^\top\big(A_N^\top P_N + P_NA_N\big)x_N + 2x_N^\top P_N(B_Nu_N + C_Nw_N) + 2x_N^\top P_N\sum_{i=1}^{N-1}D_{N,i}x_i.\end{aligned}$$

With the notation of (7), we have $\sum_{i=1}^{N-1} D_{N,i}x_i = D_{N,-}\chi$. Since $P_N \succ 0$, the Cauchy-Schwarz inequality [23] as stated in Lemma 3 yields

$x_N^\top P_N D_{N,-}\chi \leq \|x_N\|_{P_N}\|D_{N,-}\chi\|_{P_N}$ and $x_N^\top P_N(B_Nu_N + C_Nw_N) \leq \|x_N\|_{P_N}\|B_Nu_N + C_Nw_N\|_{P_N}$.

We will demonstrate the stabilizing property of the control u_N minimizing $\|B_Nu_N + C_Nw_N\|_{P_N}$ when w_N is chosen to maximize this norm. By definition these choices of u_N and w_N yield $\|B_Nu_N + C_Nw_N\|_{P_N} \leq z_{max}^{P_N}$. Then,

$$\frac{d}{dt}\|x_N(t)\|_{P_N}^2 \leq -x_N^\top Q_N x_N + 2\|x_N\|_{P_N}\left(z_{max}^{P_N} + \|D_{N,-}\chi\|_{P_N}\right).$$

Since $Q_N \succ 0$, we have $-x_N^\top Q_N x_N \leq -\lambda_{min}^{Q_N} x_N^\top x_N$ [27] and $\|x_N\|_{P_N}^2 \leq \lambda_{max}^{P_N} x_N^\top x_N$ leads to $-x_N^\top x_N \leq \frac{-1}{\lambda_{max}^{P_N}}\|x_N\|_{P_N}^2$. Hence, we obtain

$$\begin{aligned}\frac{d}{dt}\|x_N(t)\|_{P_N}^2 &\leq -\frac{\lambda_{min}^{Q_N}}{\lambda_{max}^{P_N}}\|x_N\|_{P_N}^2 + 2\|x_N\|_{P_N}\left(z_{max}^{P_N} + \|D_{N,-}\chi\|_{P_N}\right) \\ &\leq -2\alpha_N\|x_N(t)\|_{P_N}^2 + 2\beta_N(t)\|x_N(t)\|_{P_N},\end{aligned}$$

by definition of α_N and β_N. We introduce $y_N(t) := \|x_N(t)\|_{P_N}$, so that we have

$$\frac{d}{dt}y_N^2(t) = 2y_N(t)\dot{y}_N(t) \leq -2\alpha_N y_N(t)^2 + 2\beta_N(t)y_N(t).$$

For $y_N(t) > 0$, we then have $\dot{y}_N(t) \leq -\alpha_N y_N(t) + \beta_N(t)$.

We now introduce the function $f_N(t,s) := -\alpha_N s + \beta_N(t)$. The solution of the differential equation $\dot{s}(t) = f_N\big(t, s(t)\big)$, $s(0) = \|x_N(0)\|_{P_N}$ is $s(t) = e^{-\alpha_N t}\Big(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N\tau}\beta_N(\tau)\,d\tau\Big)$. Since $f_N(t,s)$ is Lipschitz in s and continuous in t, $\dot{y}_N(t) \leq f_N\big(t, y_N(t)\big)$ and $y_N(0) = z(0)$, the Comparison Lemma of [27] states that $y_N(t) \leq s(t)$ for all $t \geq 0$, hence (11) holds. □

Since A_N is Hurwitz, we can bound the steady-state value of the state x_N despite undesirable inputs that cannot be counteracted. We will now study the impact of x_N on the rest of the network, whose dynamics follow (6). Recall that $\chi = \big(x_1, \ldots, x_{N-1}\big)$ and $\hat{D}$ was defined in (7). Dynamics (6) are perturbed by the term $D_{-,N}x_N(t)$ bounded in Proposition 7. We can then evaluate how term $D_{-,N}x_N(t)$ impacts $\chi(t)$ by building on Proposition 7 and reusing P_N, Q_N, α_N, β_N and $z_{max}^{P_N}$. We will first investigate the scenario where $\hat{B}$ is full rank before requiring only controllability of pair $(\hat{A}+\hat{D}, \hat{B})$.

4.1 Fully-actuated networks

In this section we assume that the combined control matrix of the first $N-1$ subsystems $\hat{B}$ is full rank.

Proposition 8. *If $\hat{A}+\hat{D}$ and A_N are Hurwitz, $\hat{B}$ is full rank, and $C_N\mathcal{W}_N \not\subseteq B_N\mathcal{U}_N$, then for any $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A}+\hat{D})^\top\hat{P} + \hat{P}(\hat{A}+\hat{D}) = -\hat{Q}$ let us define the constants $b_{min}^{\hat{P}} := \min\limits_{\hat{u}\,\in\,\partial\hat{\mathcal{U}}}\big\{\|\hat{B}\hat{u}\|_{\hat{P}}\big\}$, $\alpha := \frac{\lambda_{min}^{\hat{Q}}}{2\lambda_{max}^{\hat{P}}}$, $\gamma := \sqrt{\frac{\max\lambda(D_{-,N}^\top\hat{P}D_{-,N})}{\lambda_{min}^{P_N}}}$ and $\gamma_N := \sqrt{\frac{\max\lambda(D_{N,-}^\top P_N D_{N,-})}{\lambda_{min}^{\hat{P}}}}$. If $\alpha\alpha_N \neq \gamma\gamma_N$, then there exist $h_\pm \in \mathbb{R}$ and $r_\pm \in \mathbb{R}$ such that*

$$\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N} + h_+e^{(r_+-\alpha_N)t} + h_-e^{(r_--\alpha_N)t}\right\}. \tag{12}$$

If $\alpha\alpha_N = \gamma\gamma_N$, there are constants $h_\pm \in \mathbb{R}$ such that

$$\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha + \alpha_N}t + h_+ + h_-e^{-(\alpha+\alpha_N)t}\right\}. \tag{13}$$

Proof. Since $\hat{A}+\hat{D}$ is Hurwitz, there exist a symmetric $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A}+\hat{D})^\top\hat{P} + \hat{P}(\hat{A}+\hat{D}) = -\hat{Q}$ according to Lyapunov theory [26]. Following the same steps as in the proof of Proposition 7 with $\chi^\top\hat{P}\chi = \|\chi\|_{\hat{P}}^2$ and χ following the dynamics (6), we first obtain

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 = \chi^\top\big((\hat{A}+\hat{D})^\top\hat{P} + \hat{P}(\hat{A}+\hat{D})\big)\chi + 2\chi^\top\hat{P}\hat{B}\hat{u} + 2\chi^\top\hat{P}D_{-,N}x_N.$$

Because $\hat{B}$ is full rank, for all $\chi(t) \neq 0$ there exist $\hat{u}(t) \in \hat{\mathcal{U}}$ such that $\hat{B}\hat{u}(t) = -\frac{\chi(t)}{\|\chi(t)\|_{\hat{P}}}b_{min}^{\hat{P}}$, as shown in the proof of Proposition 3 of [17]. Then,

$$\chi(t)^\top\hat{P}\hat{B}\hat{u}(t) = \frac{-\chi(t)^\top\hat{P}\chi(t)}{\|\chi(t)\|_{\hat{P}}}b_{min}^{\hat{P}} = -\|\chi(t)\|_{\hat{P}}b_{min}^{\hat{P}}.$$

Since $\|\cdot\|_{\hat{P}}$ is a norm, it verifies the Cauchy-Schwarz inequality [23] $\chi^\top\hat{P}D_{-,N}x_N \leq \|\chi\|_{\hat{P}}\|D_{-,N}x_N\|_{\hat{P}}$. Then,

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 \leq -\chi(t)^\top\hat{Q}\chi(t) - 2\|\chi(t)\|_{\hat{P}}b_{min}^{\hat{P}} + 2\|\chi(t)\|_{\hat{P}}\|D_{-,N}x_N(t)\|_{\hat{P}}.$$

Because $\hat{P} \succ 0$ and $\hat{Q} \succ 0$, we obtain $-\chi^\top\hat{Q}\chi \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}\|\chi\|_{\hat{P}}^2$. Since $\hat{P}^\top = \hat{P} \succ 0$ and $\hat{P}_N \succ 0$, Lemma 2 states $\|D_{-,N}x_N\|_{\hat{P}} \leq \gamma\|x_N\|_{P_N}$. We now combine these inequalities into

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}\|\chi(t)\|_{\hat{P}}^2 + 2\|\chi(t)\|_{\hat{P}}\left(\gamma\|x_N(t)\|_{P_N} - b_{min}^{\hat{P}}\right).$$

Following Proposition 7, we also include bound (11) on $\|x_N(t)\|_{P_N}$, which yields

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}\|\chi(t)\|_{\hat{P}}^2 + 2\|\chi(t)\|_{\hat{P}}\left(\gamma e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N\tau}\beta_N(\tau)d\tau\right) - b_{min}^{\hat{P}}\right).$$

Define $y(t) := \|\chi(t)\|_{\hat{P}}$ and $y_N(t) := \|x_N(t)\|_{P_N}$. Since $P_N^\top = P_N \succ 0$ and $\hat{P} \succ 0$, Lemma 2 states

$\|D_{N,-}\chi\|_{P_N} \le \gamma_N \|\chi\|_{\hat{P}}$, which can be used in β_N defined in Proposition 7 as

$$\beta_N(\tau) = z_{max}^{P_N} + \|D_{N,-}\chi(\tau)\|_{P_N} \le z_{max}^{P_N} + \gamma_N \|\chi(\tau)\|_{\hat{P}} = z_{max}^{P_N} + \gamma_N y(\tau). \tag{14}$$

We notice that $\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 = 2y(t)\dot{y}(t)$, which yields

$$2y(t)\dot{y}(t) \le -2\alpha y(t)^2 + 2y(t)\left(\gamma y_N(0)e^{-\alpha_N t} + \gamma e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau}\left(z_{max}^{P_N} + \gamma_N y(\tau)\right)d\tau - b_{min}^{\hat{P}}\right).$$

For $y(t) > 0$ we can divide both sides of the inequality by $2y(t)$ and we calculate the following trivial integral

$$e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau}\, d\tau = e^{-\alpha_N t}\frac{e^{\alpha_N t} - 1}{\alpha_N} = \frac{1 - e^{-\alpha_N t}}{\alpha_N},$$

so that the differential inequality becomes

$$\begin{aligned}
\dot{y}(t) &\le -\alpha y(t) + \gamma y_N(0)e^{-\alpha_N t} + \gamma z_{max}^{P_N}\frac{1 - e^{-\alpha_N t}}{\alpha_N} + \gamma\gamma_N e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau} y(\tau)\, d\tau - b_{min}^{\hat{P}}, \\
&\le -\alpha y(t) + \frac{\gamma z_{max}^{P_N}}{\alpha_N} - b_{min}^{\hat{P}} + \gamma\left(y_N(0) - \frac{z_{max}^{P_N}}{\alpha_N}\right)e^{-\alpha_N t} + \gamma\gamma_N e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau} y(\tau)\, d\tau.
\end{aligned}$$

Now multiply both sides by $e^{\alpha_N t} > 0$ and define $v(t) = e^{\alpha_N t} y(t)$. Then, $\dot{v}(t) = \alpha_N v(t) + e^{\alpha_N t}\dot{y}(t)$, which leads to

$$e^{\alpha_N t}\dot{y}(t) = \dot{v}(t) - \alpha_N v(t) \le -\alpha v(t) + \left(\frac{\gamma z_{max}^{P_N}}{\alpha_N} - b_{min}^{\hat{P}}\right)e^{\alpha_N t} + \gamma\left(y_N(0) - \frac{z_{max}^{P_N}}{\alpha_N}\right) + \gamma\gamma_N\int_0^t v(\tau)\, d\tau.$$

We introduce the function

$$f\big(t, s(t)\big) := (\alpha_N - \alpha)s(t) + \left(\frac{\gamma z_{max}^{P_N}}{\alpha_N} - b_{min}^{\hat{P}}\right)e^{\alpha_N t} + \gamma\left(y_N(0) - \frac{z_{max}^{P_N}}{\alpha_N}\right) + \gamma\gamma_N\int_0^t s(\tau)\, d\tau,$$

so that $\dot{v}(t) \le f\big(t, v(t)\big)$. Now we search for a solution to the differential equation $\dot{s}(t) = f\big(t, s(t)\big)$. Differentiating this equation yields

$$\ddot{s}(t) = \frac{d}{dt}f\big(t, s(t)\big) = (\alpha_N - \alpha)\dot{s}(t) + \left(\frac{\gamma z_{max}^{P_N}}{\alpha_N} - b_{min}^{\hat{P}}\right)\alpha_N e^{\alpha_N t} + 0 + \gamma\gamma_N s(t),$$

i.e.,

$$\ddot{s}(t) + (\alpha - \alpha_N)\dot{s}(t) - \gamma\gamma_N s(t) - \left(\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}\right)e^{\alpha_N t} = 0. \tag{15}$$

We need to distinguish two cases when solving this differential equation as detailed in Lemma 5. If $\alpha\alpha_N \ne \gamma\gamma_N$, then the solution of (15) is

$$s(t) = pe^{\alpha_N t} + h_+ e^{r_+ t} + h_- e^{r_- t}$$

with

$$p = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N}, \qquad r_\pm = \frac{1}{2}\Big(\alpha_N - \alpha \pm \sqrt{(\alpha - \alpha_N)^2 + 4\gamma\gamma_N}\Big), \tag{16}$$

and $h_\pm \in \mathbb{R}$ are two constants to be determined. Now we can apply the Comparison Lemma of [27] stating that if $\dot{s}(t) = f\big(t, s(t)\big)$, f is continuous in t and locally Lipschitz in s and $s(0) = v(0)$, then $\dot{v}(t) \le f\big(t, v(t)\big)$ implies $v(t) \le s(t)$ for all $t \ge 0$. Using $\|\chi(t)\|_{\hat{P}} = y(t) = e^{-\alpha_N t}v(t) \le e^{-\alpha_N t}s(t)$, we finally obtain (12). To determine $h_\pm$, we use the initial conditions $s(0) = v(0) = y(0)$ and $\dot{s}(0) = f\big(0, s(0)\big)$, which yield

$$h_\pm = \frac{(\alpha_N - \alpha - r_\mp)\|\chi(0)\|_{\hat{P}} + \gamma\|x_N(0)\|_{P_N} - b_{min}^{\hat{P}} + (r_\mp - \alpha_N)p}{\pm\sqrt{(\alpha - \alpha_N)^2 + 4\gamma\gamma_N}}.$$

In the case $\alpha\alpha_N = \gamma\gamma_N$, the solution of (15) is

$$s(t) = pte^{\alpha_N t} + h_+ e^{\alpha_N t} + h_- e^{-\alpha t}$$

with

$$p = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha + \alpha_N}, \quad \text{and} \quad h_\pm = \frac{\frac{1}{2}\big(- \alpha_N - \alpha \pm 3(\alpha - \alpha_N)\big)\|\chi(0)\|_{\hat{P}} \mp \gamma \|x_N(0)\|_{P_N} \pm b_{min}^{\hat{P}} \pm p}{\alpha_N + \alpha},$$

obtained from the initial conditions $s(0) = y(0)$ and $\dot{s}(0) = f\big(0, s(0)\big)$. Applying the Comparison Lemma of [27] as above, we obtain $\|\chi(t)\|_{\hat{P}} = y(t) = e^{-\alpha_N t} v(t) \leq e^{-\alpha_N t} s(t)$, which yields (13). □

We can now derive conditions for subsystem (6) to be resiliently stabilizable despite the perturbations created by x_N. These conditions solve Problem 3 in the fully-actuated network scenario.

Theorem 4. *If $\hat{A} + \hat{D}$ and A_N are Hurwitz, $\hat{B}$ is full rank, and $C_N \mathcal{W}_N \nsubseteq B_N \mathcal{U}_N$, $\gamma\gamma_N \leq \alpha\alpha_N$ and $\gamma z_{max}^{P_N} < \alpha_N b_{min}^{\hat{P}}$, then subsystem* (6) *is resiliently stabilizable in finite time.*

Proof. Let us first consider the case $\gamma\gamma_N = \alpha\alpha_N$. Since $\alpha > 0$ and $\alpha_N > 0$, the exponential term in (13) goes to zero asymptotically. By assumption $\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}} < 0$ and $\alpha + \alpha_N > 0$, so the ratio of these factors is negative. Because this ratio is multiplied by t in (13), there exists some time $T \geq 0$ such that for all $t \geq T$

$$\frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha + \alpha_N} t + h_+ + h_- e^{-(\alpha+\alpha_N)t} \leq 0. \tag{17}$$

Therefore, according to (13), subsystem (6) is resiliently stabilizable in finite time.

Now consider the case $\gamma\gamma_N < \alpha\alpha_N$. We will show that this inequality is equivalent to $r_+ - \alpha_N < 0$, where r_+ is defined in (16). Indeed,

$$\begin{aligned} r_+ - \alpha_N < 0 \iff & -\frac{1}{2}(\alpha_N + \alpha) + \frac{1}{2}\sqrt{(\alpha - \alpha_N)^2 + 4\gamma\gamma_N} < 0 \iff (\alpha - \alpha_N)^2 + 4\gamma\gamma_N < (\alpha_N + \alpha)^2 \\ \iff & -2\alpha\alpha_N + 4\gamma\gamma_N < 2\alpha\alpha_N \iff \gamma\gamma_N < \alpha\alpha_N. \end{aligned}$$

Since $r_- \leq r_+$, we also have $r_- - \alpha_N < 0$, so both exponential terms in (12) converge to zero. Additionally, the fraction term in (12) is negative, so the right-hand side of (12) reaches zero in finite time. Therefore, subsystem (6) is resiliently stabilizable in finite time. □

Let us now give some intuition concerning Theorem 4. Since γ is proportional to the norm of the matrix $D_{-,N}$ which multiplies $x_N(t)$ in (6), γ quantifies the impact of nonresilient subsystem (4) of state $x_N(t)$ on the rest of the network (6) of state $\chi(t)$. Reciprocally, γ_N quantifies the impact of $\chi(t)$ on $x_N(t)$. On the other hand, $\alpha = \frac{\lambda_{min}^{\hat{Q}}}{2\lambda_{max}^{\hat{P}}}$ relates to the joint stability of the first $N-1$ subsystems of network (6), while α_N relates to the stability of malfunctioning subsystem (4). Therefore, condition $\gamma\gamma_N \leq \alpha\alpha_N$ follows the intuition that the magnitude of the perturbations arising from the coupling between subsystems (6) and (4) must be weaker than the stability of each of these subsystems.

We will now discuss the other stabilizability condition of Theorem 4, namely $\gamma z_{max}^{P_N} < \alpha_N b_{min}^{\hat{P}}$. Since $z_{max}^{P_N}$ describes the magnitude of the destabilizing inputs in subsystem (4), term $\gamma z_{max}^{P_N}$ quantifies the destabilizing influence of w_N on the state of the rest of the network χ. On the other hand, $b_{min}^{\hat{P}}$ relates to the magnitude of the stabilizing inputs in subsystem (6) and α_N relates to the Hurwitzness of malfunctioning subsystem (4). Therefore, condition $\gamma z_{max}^{P_N} < \alpha_N b_{min}^{\hat{P}}$ carries the intuition that the stabilizing terms of the network must overcome the destabilizing ones.

Since we have bounded the state χ of the first $N-1$ subsystems, we can now derive a closed-form bound on the state x_N of the malfunctioning subsystem N. Indeed, the bound on x_N derived in Proposition 7 depends on $\chi(t)$ through the term $\beta_N(t)$.

Proposition 9. *If $\hat{A}+\hat{D}$ and A_N are Hurwitz, $\hat{B}$ is full rank, and $C_N\mathcal{W}_N \nsubseteq B_N\mathcal{U}_N$, we can bound the state of subsystem (4) as*

$$\|x_N(t)\|_{P_N} \leq \begin{cases} \max\left\{0,\ e^{-\alpha_N t}\|x_N(0)\|_{P_N} + M(t)\right\} & \text{if} \quad \|\chi(t)\|_{\hat{P}} > 0, \\ \frac{z_{max}^{P_N}}{\alpha_N} + \left(\|x_N(0)\|_{P_N} - \frac{z_{max}^{P_N}}{\alpha_N}\right) e^{-\alpha_N t} & \text{otherwise,} \end{cases} \tag{18}$$

with

$$M(t) = \begin{cases} \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N}\left(1-e^{-\alpha_N t}\right) + e^{-\alpha_N t}\left(\frac{\gamma_N h_+}{r_+}\left(e^{r_+ t}-1\right) + \frac{\gamma_N h_-}{r_-}\left(e^{r_- t}-1\right)\right) & \text{if } \alpha\alpha_N \neq \gamma\gamma_N, \\ \frac{1-e^{-\alpha_N t}}{\alpha_N}\left(\gamma_N h_+ + \frac{\alpha_N z_{max}^{P_N} + \gamma_N b_{min}^{\hat{P}}}{\alpha+\alpha_N}\right) + \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha+\alpha_N} t + \frac{\gamma_N h_-}{\alpha}\left(1-e^{-\alpha t}\right)e^{-\alpha_N t} & \text{otherwise.} \end{cases} \tag{19}$$

Proof. We recall from Proposition 7 that $\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N \tau}\beta_N(\tau)\,d\tau\right)$ (11). Following (14), we have $\beta_N(t) \leq z_{max}^{P_N} + \gamma_N\|\chi(t)\|_{\hat{P}}$. We can bound $\|\chi(t)\|_{\hat{P}}$ with (12) or (13) from Proposition 8 depending on the values of $\alpha\alpha_N$ and $\gamma\gamma_N$.

We start with the case where $\alpha\alpha_N \neq \gamma\gamma_N$ and $\|\chi(t)\|_{\hat{P}} > 0$. Then, bound (12) combined with (14) yields

$$\begin{aligned} \int_0^t e^{\alpha_N\tau}\beta_N(\tau)\,d\tau &\leq \int_0^t e^{\alpha_N\tau}\left(z_{max}^{P_N} + \gamma_N p + \gamma_N h_+ e^{(r_+-\alpha_N)\tau} + \gamma_N h_- e^{(r_--\alpha_N)\tau}\right)d\tau \\ &= \frac{e^{\alpha_N t}-1}{\alpha_N}\left(z_{max}^{P_N} + \gamma_N p\right) + \frac{\gamma_N h_+}{r_+}\left(e^{r_+ t}-1\right) + \frac{\gamma_N h_-}{r_-}\left(e^{r_- t}-1\right). \end{aligned}$$

We replace p in

$$\frac{z_{max}^{P_N} + \gamma_N p}{\alpha_N} = \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N} \qquad \text{with} \qquad p = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N}.$$

Then, plugging the integral calculated above in (11), we obtain

$$\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N}\left(e^{\alpha_N t}-1\right) + \frac{\gamma_N h_+}{r_+}\left(e^{r_+ t}-1\right) + \frac{\gamma_N h_-}{r_-}\left(e^{r_- t}-1\right)\right),$$

which yields (18).

When $\|\chi(t)\|_{\hat{P}} = 0$, bound (12) is modified and yields

$$\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N\tau} z_{max}^{P_N}\,d\tau\right) = \frac{z_{max}^{P_N}}{\alpha_N} + \left(\|x_N(0)\|_{P_N} - \frac{z_{max}^{P_N}}{\alpha_N}\right)e^{-\alpha_N t}.$$

We can now address the other case where $\alpha\alpha_N = \gamma\gamma_N$ and $\|\chi(t)\|_{\hat{P}} > 0$ is bounded by (13), which yields

$$\begin{aligned} \int_0^t e^{\alpha_N\tau}\beta_N(\tau)\,d\tau &\leq \int_0^t e^{\alpha_N\tau}\left(z_{max}^{P_N} + \gamma_N p\tau + \gamma_N h_+ + \gamma_N h_- e^{-(\alpha+\alpha_N)\tau}\right)d\tau \\ &= \left(z_{max}^{P_N} + \gamma_N h_+\right)\int_0^t e^{\alpha_N\tau}\,d\tau + \gamma_N p\int_0^t \tau e^{\alpha_N\tau}\,d\tau + \gamma_N h_-\int_0^t e^{-\alpha\tau}\,d\tau \\ &= \left(z_{max}^{P_N} + \gamma_N h_+\right)\frac{e^{\alpha_N t}-1}{\alpha_N} + \frac{\gamma_N p}{\alpha_N}\left(\frac{1-e^{\alpha_N t}}{\alpha_N} + te^{\alpha_N t}\right) + \gamma_N h_-\frac{e^{-\alpha t}-1}{-\alpha} \\ &= \frac{e^{\alpha_N t}-1}{\alpha_N}\left(z_{max}^{P_N} + \gamma_N h_+ - \frac{\gamma_N p}{\alpha_N}\right) + \frac{\gamma_N p}{\alpha_N}te^{\alpha_N t} + \frac{\gamma_N h_-}{\alpha}\left(1-e^{-\alpha t}\right), \end{aligned}$$

where we calculated $\int_0^t \tau e^{\alpha_N \tau}\, d\tau$ using

$$\int_0^t \frac{d}{d\tau}\frac{\tau e^{\alpha_N \tau}}{\alpha_N} d\tau = \left[\frac{\tau e^{\alpha_N \tau}}{\alpha_N}\right]_0^t = \frac{t e^{\alpha_N t}}{\alpha_N} - 0 = \int_0^t \frac{e^{\alpha_N \tau}}{\alpha_N} d\tau + \int_0^t \tau e^{\alpha_N \tau} d\tau$$
$$= \frac{e^{\alpha_N t} - 1}{\alpha_N^2} + \int_0^t \tau e^{\alpha_N \tau} d\tau.$$

Then, we replace p with its definition:

$$\frac{\gamma_N p}{\alpha_N} = \frac{\gamma_N \gamma z_{max}^{P_N} - \gamma_N \alpha_N b_{min}^{\hat{P}}}{\alpha_N(\alpha + \alpha_N)} = \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha + \alpha_N} \qquad \text{thanks to} \qquad p = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha + \alpha_N}$$

and $\gamma \gamma_N = \alpha \alpha_N$. Multiplying the upper bound on $\int_0^t e^{\alpha_N \tau} \beta_N(\tau) d\tau$ calculated previously by $e^{-\alpha_N t}$ yields

$$\int_0^t e^{\alpha_N(\tau - t)} \beta_N(\tau) d\tau \leq \frac{1 - e^{-\alpha_N t}}{\alpha_N}\left(z_{max}^{P_N} + \gamma_N h_+ - \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha + \alpha_N}\right) + \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha + \alpha_N} t$$
$$+ \frac{\gamma_N h_-}{\alpha}\big(1 - e^{-\alpha t}\big) e^{-\alpha_N t}$$
$$\leq \frac{1 - e^{-\alpha_N t}}{\alpha_N}\left(\gamma_N h_+ + \frac{\alpha_N z_{max}^{P_N} + \gamma_N b_{min}^{\hat{P}}}{\alpha + \alpha_N}\right) + \frac{\alpha z_{max}^{P_N} - \gamma_N b_{min}^{\hat{P}}}{\alpha + \alpha_N} t + \frac{\gamma_N h_-}{\alpha}\big(1 - e^{-\alpha t}\big) e^{-\alpha_N t}$$

We finally obtain (18) thanks to (11). □

Remark 1. *The switch between bounds* (18) *is likely to be discontinuous. Indeed, the bound in* (18) *valid for* $\|\chi(t)\|_{\hat{P}} > 0$ *relies on all the overapproximations of bound* (12)*, whereas the case* $\|\chi(t)\|_{\hat{P}} = 0$ *is derived without these overapproximations.*

Thanks to Propositions 8 and 9, we now have a complete description of the network state after a nonresilient loss of control authority. These two results relied on the full rank assumption of $\hat{B}$, the control matrix of the unaffected part of the network. Because this assumption might be too restrictive, we will now employ a different approach to bound the states of an underactuated network.

4.2 Underactuated networks

Let us now assume that $\hat{B}$ is not full rank, which prevents the use of Proposition 8. Instead of the stabilizing control input of constant magnitude $\hat{B}\hat{u}(t) = -\frac{\chi(t)}{\|\chi(t)\|_{\hat{P}}} b_{min}^{\hat{P}}$ used in Proposition 8, we will employ a linear control to stabilize network state χ.

If pair $\big(\hat{A} + \hat{D}, \hat{B}\big)$ is controllable, there exist a matrix K such that $\hat{A} + \hat{D} - \hat{B}K$ is Hurwitz. Then, for any $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A} + \hat{D} - \hat{B}K)^\top \hat{P} + \hat{P}(\hat{A} + \hat{D} - \hat{B}K) = -\hat{Q}$, we can define the same constants as in Proposition 8, namely $\alpha = \frac{\lambda_{min}^{\hat{Q}}}{2\lambda_{max}^{\hat{P}}}$, $\gamma = \sqrt{\frac{\lambda_{max}^{D_{-,N}^\top \hat{P} D_{-,N}}}{\lambda_{min}^{P_N}}}$ and $\gamma_N = \sqrt{\frac{\lambda_{max}^{D_{N,-}^\top P_N D_{N,-}}}{\lambda_{min}^{\hat{P}}}}$.

Proposition 10. *If pair* $\big(\hat{A} + \hat{D}, \hat{B}\big)$ *is controllable,* A_N *is Hurwitz,* $C_N \mathcal{W}_N \nsubseteq B_N \mathcal{U}_N$*,* $\gamma \gamma_N < \alpha \alpha_N$ *and* $\sup\limits_{t \geq 0} b(t) \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$*, then* $\|\chi(t)\|_{\hat{P}} \leq \max\{0,\ b(t)\}$ *for all* $t \geq 0$*, with*

$$b(t) := p + h_+ e^{(r_+ - \alpha_N)t} + h_- e^{(r_- - \alpha_N)t}, \tag{20}$$

$$h_{\pm} = \frac{(\alpha_N - \alpha - r_{\mp})\|\chi(0)\|_{\hat{P}} + \gamma\|x_N(0)\|_{P_N} + (r_{\mp} - \alpha_N)p}{\pm\sqrt{(\alpha_N - \alpha)^2 + 4\gamma\gamma_N}} \quad \textit{and} \quad p = \frac{\gamma z_{max}^{P_N}}{\alpha\alpha_N - \gamma\gamma_N}.$$

Proof. Since pair $\big(\hat{A}+\hat{D}, \hat{B}\big)$ is controllable, there exists a matrix K such that $\hat{A}+\hat{D}-\hat{B}K$ is Hurwitz [27]. Then, there exists $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A}+\hat{D}-\hat{B}K)^\top \hat{P} + \hat{P}(\hat{A}+\hat{D}-\hat{B}K) = -\hat{Q}$ according to Lyapunov theory [26]. We will follow the same steps as in the proof of Proposition 8 with $\chi^\top \hat{P}\chi = \|\chi\|_{\hat{P}}^2$ and χ following the dynamics (6) with $\hat{u}(t) = -K\chi(t)$. Once we obtain bounds on $\chi(t)$ we will verify under which conditions is $\hat{u}$ admissible. We first obtain

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 = \chi(t)^\top \big((\hat{A}+\hat{D}-\hat{B}K)^\top \hat{P} + \hat{P}(\hat{A}+\hat{D}-\hat{B}K)\big)\chi(t) + 2\chi(t)^\top \hat{P} D_{-,N} x_N(t).$$

We then proceed as in Proposition 8, but without the term $b_{min}^{\hat{P}}$. Since $\|\cdot\|_{\hat{P}}$ is a norm, it verifies the Cauchy-Schwarz inequality [23] $\chi^\top \hat{P} D_{-,N} x_N \leq \|\chi(t)\|_{\hat{P}} \|D_{-,N} x_N(t)\|_{\hat{P}}$. Then,

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 \leq -\chi(t)^\top \hat{Q}\chi(t) + 2\|\chi(t)\|_{\hat{P}} \|D_{-,N} x_N(t)\|_{\hat{P}}.$$

Because $\hat{P} \succ 0$ and $\hat{Q} \succ 0$, we obtain $-\chi^\top \hat{Q}\chi \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}\|\chi\|_{\hat{P}}^2$. Since $\hat{P}^\top = \hat{P} \succ 0$ and $\hat{P}_N \succ 0$, Lemma 2 states $\|D_{-,N} x_N\|_{\hat{P}} \leq \gamma \|x_N\|_{P_N}$. We now combine these inequalities into

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}\|\chi(t)\|_{\hat{P}}^2 + 2\|\chi(t)\|_{\hat{P}}\gamma\|x_N(t)\|_{P_N}.$$

Following Proposition 7, we also include bound (11) on $\|x_N(t)\|_{P_N}$, which yields

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}\|\chi(t)\|_{\hat{P}}^2 + 2\|\chi(t)\|_{\hat{P}}\gamma e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N \tau}\big(z_{max}^{P_N} + \|D_{N,-}\chi(\tau)\|_{P_N}\big)d\tau\right).$$

Since $P_N^\top = P_N \succ 0$ and $\hat{P} \succ 0$, Lemma 2 states $\|D_{N,-}\chi\|_{P_N} \leq \gamma_N \|\chi\|_{\hat{P}}$. Define $y(t) := \|\chi(t)\|_{\hat{P}}$ and $y_N(t) := \|x_N(t)\|_{P_N}$. We notice that $\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 = 2y(t)\dot{y}(t)$, which yields

$$2y(t)\dot{y}(t) \leq -2\alpha y(t)^2 + 2y(t)\left(\gamma y_N(0)e^{-\alpha_N t} + \gamma e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau}\big(z_{max}^{P_N} + \gamma_N y(\tau)\big)\, d\tau\right).$$

For $y(t) > 0$ we can divide both sides of the inequality by $2y(t)$ so that the differential inequality becomes

$$\begin{aligned}\dot{y}(t) &\leq -\alpha y(t) + \gamma y_N(0)e^{-\alpha_N t} + \gamma z_{max}^{P_N}\frac{1 - e^{-\alpha_N t}}{\alpha_N} + \gamma\gamma_N e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau} y(\tau)\, d\tau, \\ &\leq -\alpha y(t) + \frac{\gamma z_{max}^{P_N}}{\alpha_N} + \gamma\left(y_N(0) - \frac{z_{max}^{P_N}}{\alpha_N}\right)e^{-\alpha_N t} + \gamma\gamma_N e^{-\alpha_N t}\int_0^t e^{\alpha_N \tau} y(\tau)\, d\tau.\end{aligned}$$

Now multiply both sides by $e^{\alpha_N t} > 0$ and define $v(t) = e^{\alpha_N t}y(t)$. Then, $\dot{v}(t) = \alpha_N v(t) + e^{\alpha_N t}\dot{y}(t)$, which leads to

$$e^{\alpha_N t}\dot{y}(t) = \dot{v}(t) - \alpha_N v(t) \leq -\alpha v(t) + \frac{\gamma z_{max}^{P_N}}{\alpha_N}e^{\alpha_N t} + \gamma\left(y_N(0) - \frac{z_{max}^{P_N}}{\alpha_N}\right) + \gamma\gamma_N\int_0^t v(\tau)\, d\tau.$$

We introduce the function

$$g\big(t, s(t)\big) := (\alpha_N - \alpha)s(t) + \frac{\gamma z_{max}^{P_N}}{\alpha_N}e^{\alpha_N t} + \gamma\left(y_N(0) - \frac{z_{max}^{P_N}}{\alpha_N}\right) + \gamma\gamma_N\int_0^t s(\tau)\, d\tau,$$

so that $\dot{v}(t) \leq g\big(t, v(t)\big)$. Now we search for a solution to the differential equation $\dot{s}(t) = g\big(t, s(t)\big)$. Differentiating this equation yields

$$\ddot{s}(t) = \frac{d}{dt}g\big(t, s(t)\big) = (\alpha_N - \alpha)\dot{s}(t) + \gamma z_{max}^{P_N}e^{\alpha_N t} + 0 + \gamma\gamma_N s(t).$$

If $\alpha\alpha_N \neq \gamma\gamma_N$, then the solution to this differential equation is $s(t) = pe^{\alpha_N t} + h_+ e^{r_+ t} + h_- e^{r_- t}$, with

$r_\pm = \frac{1}{2}\big(\alpha_N - \alpha \pm \sqrt{(\alpha-\alpha_N)^2 + 4\gamma\gamma_N}\big)$, $p = \frac{\gamma z_{max}^{P_N}}{\alpha\alpha_N - \gamma\gamma_N} > 0$ and two constants $h_\pm \in \mathbb{R}$. Using the Comparison Lemma of [27] we obtain

$$\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ p + h_+ e^{(r_+ - \alpha_N)t} + h_- e^{(r_- - \alpha_N)t}\right\}, \tag{21}$$

for all $t \geq 0$ and the initial conditions are $s(0) = y(0)$ and $\dot{s}(0) = g(0, s(0))$, i.e.,

$$p + h_+ + h_- = \|\chi(0)\|_{\hat{P}} \quad \text{and} \quad \alpha_N p + h_+ r_+ + h_- r_- = (\alpha_N - \alpha)\|\chi(0)\|_{\hat{P}} + \gamma\|x_N(0)\|_{P_N}.$$

These initial conditions can be solved to determine $h_\pm$.

Bound (21) is only valid when $\hat{u}(t) = -K\chi(t) \in \mathcal{U} = [-1,1]^m$. For this control law to be admissible, we then need $\|\chi(t)\|_{\hat{P}} \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$ at all times $t \geq 0$, since $\|\hat{u}(t)\| \leq \|K\|\|\chi(t)\| \leq \|K\|\frac{\|\chi(t)\|_{\hat{P}}}{\sqrt{\lambda_{min}^{\hat{P}}}}$.

Since $\|\chi(t)\|_{\hat{P}} \leq \sup\limits_{t\geq 0} b(t) \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$, $\hat{u}$ is admissible. □

Remark 2. *The condition $\gamma\gamma_N < \alpha\alpha_N$ in Proposition 10 is necessary for the boundedness of $\chi(t)$, which in turn guarantees the admissibility of the control law $\hat{u}(t) = -K\chi(t)$.*

Indeed, if $\gamma\gamma_N > \alpha\alpha_N$, then $r_+ - \alpha_N > 0$, which leads to the divergence of the corresponding exponential term in (21) and hence $\chi(t)$ might not be bounded.

On the other hand, if $\alpha\alpha_N = \gamma\gamma_N$, the same process as in Proposition 10 leads to $\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ \frac{\gamma z_{max}^{P_N}}{\alpha+\alpha_N}t + h_+ + h_- e^{-(\alpha+\alpha_N)t}\right\}$ for all $t \geq 0$, for some constants $h_\pm \in \mathbb{R}$. The term linear in t grows unbounded since $\gamma z_{max}^{P_N} > 0$ and $\alpha + \alpha_N > 0$. In this case, $\chi(t)$ might not be bounded.

Note that the perturbation from subsystem (4) in norm bounds (21) is modeled by term $z_{max}^{P_N} > 0$ of constant magnitude. Hence, this perturbation cannot be overcome by the linear control $\hat{u}(t) = -K\chi(t)$ when χ is near 0. That is why Proposition 10 only guarantees the boundedness of χ and not its resilient stabilizability.

Proposition 10 also requires the calculation of the supremum of $b(t)$, defined in (20). We perform this calculation with Algorithm 1 by determining the only stationary point of b. The process of Algorithm 1 is justified in detail by Lemma 6.

Algorithm 1: Computation of the supremum of $b(t)$ from (20)

if $\gamma\gamma_N < \alpha\alpha_N$ **then**
- $T = \frac{1}{r_- - r_+}\ln\left(\frac{h_+(r_+ - \alpha_N)}{h_-(\alpha_N - r_-)}\right)$ `# extremal of` b
- **if** $\ddot{b}(T) > 0$ **then**
  - $\sup\{b(t) : t \geq 0\} = \max\{p + h_+ + h_-,\ p\}$
- **else**
  - $\sup\{b(t) : t \geq 0\} = b(T)$
- **end**

end

Remark 3. *If the network is initially at rest when the loss of control authority occurs, i.e., if $\chi(0) = 0$ and $x_N(0) = 0$, then $h_+ < 0$ and $h_- > 0$, so that $m = p + h_- = -h_+ = \frac{-(r_- + \alpha_N)\gamma z_{max}^{P_N}}{(\alpha\alpha_N - \gamma\gamma_N)\sqrt{(\alpha_N-\alpha)^2 + 4\gamma\gamma_N}}$.*

Using bound (21) in (11), we can now derive a closed-form bound on x_N as we did in Proposition 9 when $\hat{B}$ was full rank.

Proposition 11. *If pair $\big(\hat{A}+\hat{D}, \hat{B}\big)$ is controllable, A_N is Hurwitz, $C_N\mathcal{W}_N \nsubseteq B_N\mathcal{U}_N$, $\gamma\gamma_N < \alpha\alpha_N$ and $m \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$, then*

$$\|x_N(t)\|_{P_N} \leq \begin{cases} \max\big\{0,\ e^{-\alpha_N t}\|x_N(0)\|_{P_N} + M(t)\big\} & \text{if} \quad \|\chi(t)\|_{\hat{P}} > 0, \\ \frac{z_{max}^{P_N}}{\alpha_N} + \Big(\|x_N(0)\|_{P_N} - \frac{z_{max}^{P_N}}{\alpha_N}\Big)\, e^{-\alpha_N t} & \text{otherwise,} \end{cases} \tag{22}$$

with

$$M(t) = \frac{\alpha z_{max}^{P_N}\big(1-e^{-\alpha_N t}\big)}{\alpha\alpha_N - \gamma\gamma_N} + e^{-\alpha_N t}\left(\frac{\gamma_N h_+}{r_+}\big(e^{r_+ t}-1\big) + \frac{\gamma_N h_-}{r_-}\big(e^{r_- t}-1\big)\right). \tag{23}$$

Proof. We recall from Proposition 7 that $\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\Big(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N\tau}\beta_N(\tau)\, d\tau\Big)$ (11). Following (14), we have $\beta_N(t) \leq z_{max}^{P_N} + \gamma_N\|\chi(t)\|_{\hat{P}}$ where we bound $\|\chi(t)\|_{\hat{P}}$ with (21)

$$\begin{aligned} \int_0^t e^{\alpha_N\tau}\beta_N(\tau)\, d\tau &\leq \int_0^t e^{\alpha_N\tau}\Big(z_{max}^{P_N} + \gamma_N p + \gamma_N h_+ e^{(r_+-\alpha_N)\tau} + \gamma_N h_- e^{(r_--\alpha_N)\tau}\Big) d\tau \\ &= \frac{e^{\alpha_N t}-1}{\alpha_N}\big(z_{max}^{P_N} + \gamma_N p\big) + \frac{\gamma_N h_+}{r_+}\big(e^{r_+ t}-1\big) + \frac{\gamma_N h_-}{r_-}\big(e^{r_- t}-1\big). \end{aligned}$$

We replace p in

$$\frac{z_{max}^{P_N} + \gamma_N p}{\alpha_N} = \frac{z_{max}^{P_N}(\alpha\alpha_N - \gamma\gamma_N) + \gamma_N\gamma z_{max}^{P_N}}{\alpha_N(\alpha\alpha_N - \gamma\gamma_N)} = \frac{\alpha z_{max}^{P_N}}{\alpha\alpha_N - \gamma\gamma_N} \quad \text{with} \quad p = \frac{\gamma z_{max}^{P_N}}{\alpha\alpha_N - \gamma\gamma_N}.$$

Then, plugging the integral calculated above in (11), we obtain

$$\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \frac{\alpha z_{max}^{P_N}}{\alpha\alpha_N - \gamma\gamma_N}\big(e^{\alpha_N t}-1\big) + \frac{\gamma_N h_+}{r_+}\big(e^{r_+ t}-1\big) + \frac{\gamma_N h_-}{r_-}\big(e^{r_- t}-1\big)\right),$$

which yields (22).

When $\|\chi(t)\|_{\hat{P}} = 0$, we have $\beta_N(t) \leq z_{max}^{P_N}$, which simplifies (11) as follows

$$\|x_N(t)\|_{P_N} \leq e^{-\alpha_N t}\left(\|x_N(0)\|_{P_N} + \int_0^t e^{\alpha_N\tau} z_{max}^{P_N}\, d\tau\right) = \frac{z_{max}^{P_N}}{\alpha_N} + \left(\|x_N(0)\|_{P_N} - \frac{z_{max}^{P_N}}{\alpha_N}\right) e^{-\alpha_N t}.$$

□

Using Propositions 10 and 11, we can now quantify the effect of the loss of control authority over which the network was not resilient. Without the full rank assumption on $\bar{B}$, we cannot stabilize the network, but we provide a guaranteed bound on its state. This constitutes our solution to Problem 3 for an underactuated network.

5 Numerical examples

We will now illustrate the theory established in the preceding sections on two academic examples and on the IEEE 39-bus system [28]. All the data and codes necessary to run the simulations in this section are available on GitHub[1] .

[1] `https://github.com/Jean-BaptisteBouvier/Network-Resilience`

5.1 Fully actuated 3-component network

We start by testing the results of Section 4.1 on a simple network constituted of a nonresilient subsystem enduring a partial loss of control authority. This network of states χ_1, χ_2 and x_N follows dynamics

$$\dot{\chi}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \chi(t) + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \hat{u}(t) + \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix} \chi(t) + \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix} x_N(t), \quad \chi(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{24}$$

$$\dot{x}_N(t) = -x_N(t) + u_N(t) + 2w_N(t) + \begin{pmatrix} 0.3 & 0.3 \end{pmatrix} \chi(t), \qquad x_N(0) = 0, \tag{25}$$

with $\hat{u}(t) = \begin{pmatrix} \hat{u}_1(t) \\ \hat{u}_1(t) \end{pmatrix} \in [-1,1]^2$, $u_N(t) \in \mathcal{U}_N = [-1,1]$ and $w_N(t) \in \mathcal{W}_N = [-1,1]$. With the notation of (4) and (6), matrices A_N and $\hat{A} + \hat{D}$ are both Hurwitz, and the control matrix $\hat{B}$ is full rank, with

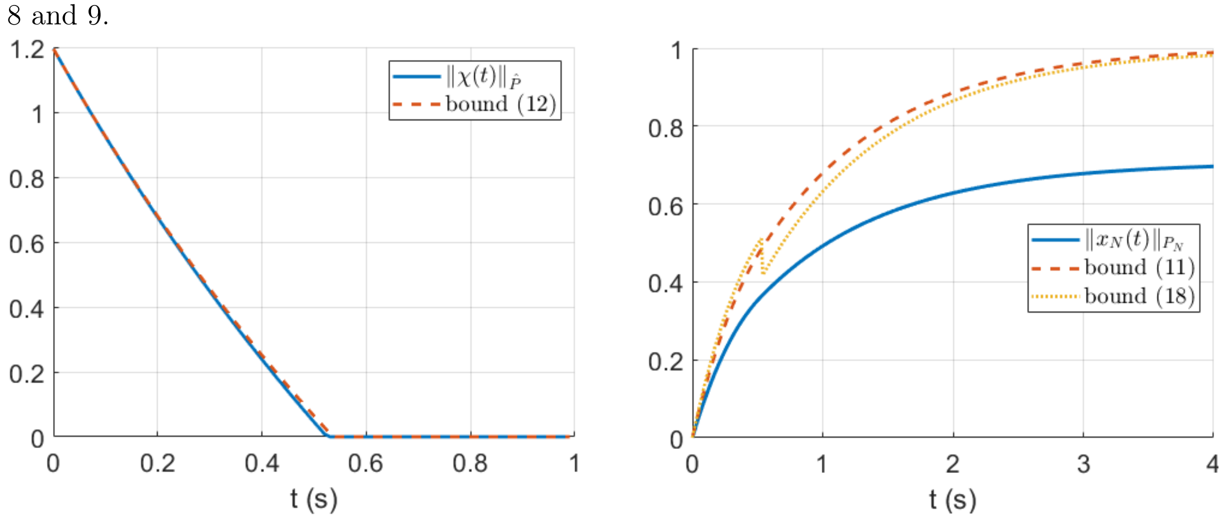
$$A_N = -1, \qquad \hat{A} + \hat{D} = \begin{pmatrix} -1 & 0.3 \\ 0.3 & -1 \end{pmatrix}, \qquad \text{and} \qquad \hat{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Additionally, $C_N \mathcal{W}_N = [-2, 2] \not\subseteq B_N \mathcal{U}_N = [-1, 1]$. Thus, all the assumptions of Propositions 7, 8 and 9 are verified. To apply these results, we solve Lyapunov equations $A_N^\top P_N + P_N A_N = -Q_N$ and $(\hat{A} + \hat{D})^\top \hat{P} + \hat{P}(\hat{A} + \hat{D}) = -\hat{Q}$ with the function *lyap* on MATLAB:

$$Q_N = 1, \quad P_N = 0.5, \qquad \hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} 0.23 & 0.05 \\ 0.05 & 0.5 \end{pmatrix}.$$

Then, following Proposition 7 $\alpha_N = 1$ and $z_{max}^{P_N} = 1$. From Proposition 8, $b_{min}^{\hat{P}} = 2$, $\alpha = 0.7$, $\gamma = 0.51$, and $\gamma_N = 0.48$.

The stabilizability conditions of Theorem 4 are satisfied since $\gamma \gamma_N = 0.25 < \alpha \alpha_N = 0.7$ and $\gamma z_{max}^{P_N} = 0.5 < \alpha_N b_{min}^{\hat{P}} = 2$. To verify that χ is indeed resiliently stabilizable in finite time by $\hat{B}\hat{u} = \frac{-\chi(t)}{\|\chi(t)\|_{\hat{P}}} b_{min}^{\hat{P}}$, we propagate $\chi(t)$ and $x_N(t)$ using control law $u_N(t) = -1$ and undesirable signal $w_N(t) = 1$. Figure 1 shows the resulting states evolution along with the bounds of Propositions 7, 8 and 9.



(a) Bounding network state $\chi(t)$ of (24).

(b) Bounding malfunctioning state $x_N(t)$ of (25).

Figure 1: Time evolution of states χ and x_N along with their bounds (11), (12) and (18).

Figure 1(a) shows the finite-time resilient stabilization of state χ and its respect of the tight bound (12). Figure 1(b) illustrates the initial divergence but overall boundedness of malfunctioning state x_N while respecting both bounds (11) and (18). As discussed in Remark 1, when $\chi(t)$ reaches

0, bound (18) operates a discontinuous switch.

5.2 Underactuated 3-component network

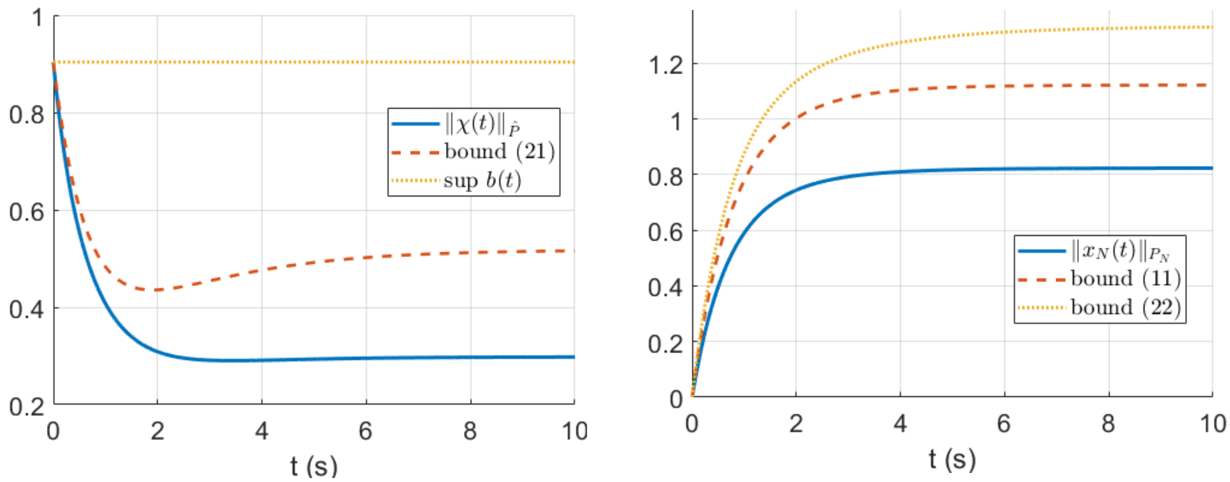
To validate the results of Section 4.2, we need $\hat{B}$ not to be full rank anymore, but the pair $(\hat{A}+\hat{D}, \hat{B})$ must remain controllable. Then, we remove the second column of $\hat{B}$ so that (24) becomes

$$\dot{\chi}(t) = \begin{pmatrix} -1 & 0.3 \\ 0.3 & -1 \end{pmatrix} \chi(t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \hat{u}(t) + \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix} x_N(t), \qquad \chi(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{26}$$

The MATLAB functions *lqr* and *lyap* choose the following gain matrix K and positive definite matrices $\hat{P}$ and $\hat{Q}$:

$$K = \begin{pmatrix} 0.6383 & 0.1521 \end{pmatrix}, \qquad \hat{P} = \begin{pmatrix} 0.22 & 0.04 \\ 0.04 & 0.5 \end{pmatrix} \qquad \text{and} \qquad \hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $\gamma\gamma_N = 0.24 < \alpha\alpha_N = 0.98$ and $\hat{u}(t) := -K\chi(t) \in [-1, 1]$. Thus, the linear feedback of Proposition 10 is admissible and its bound (21) holds as illustrated on Figure 2(a). Algorithm 1 states that bound (21) has a minimum at $T = 1.9\,s$ and its supremum occurs either at $t = 0$ or as $t \to +\infty$. Figure 2(a) validates the algorithm prediction and shows that the supremum of $b(t)$ is reached at $t = 0$.



(a) Bounding network state χ.

(b) Bounding malfunctioning state x_N.

Figure 2: Illustration of bounds (11), (21) and (22) on the states χ and x_N.

On the contrary to bound (18) on Figure 1(b), bound (22) on Figure 2(b) does not switch. Indeed, χ cannot be brought to 0 by the linear control $\hat{u}$, as explained after Proposition 10.

The bound of Proposition 11 is shown on Figure 2(b) where no switch occurs because χ cannot be brought to 0 by the linear control $\hat{u}$ as explained after Proposition 10.

As illustrated on Figure 2(b), bound (11) is tighter than bound (22). The reason for this difference in conservatism is that (11) uses directly the value of χ, while (22) replaces χ by its bound (21).

To verify the admissibility of the linear control law $\hat{u}(t) = -K\chi(t)$ we cannot use the sufficient condition of Proposition 10 as $\sup b(t)$ calculated with Algorithm 1 yields $\sup b(t) = 0.9 > \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|} = 0.71$. However, we can see on Figure 3 that $\|K\chi(t)\| \leq 1$ for all $t \geq 0$ and thus $\hat{u}$ is in fact admissible.

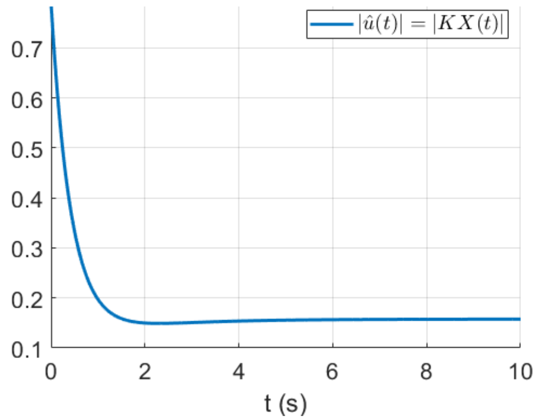


Figure 3: Linear feedback $\hat{u}(t) = -K\chi(t)$.

5.3 Resilient stabilizability of a power network

In this section, we will illustrate the results of Section 4.2 on the IEEE 39-bus system [28] linearized in [29] and represented on Figure 4.

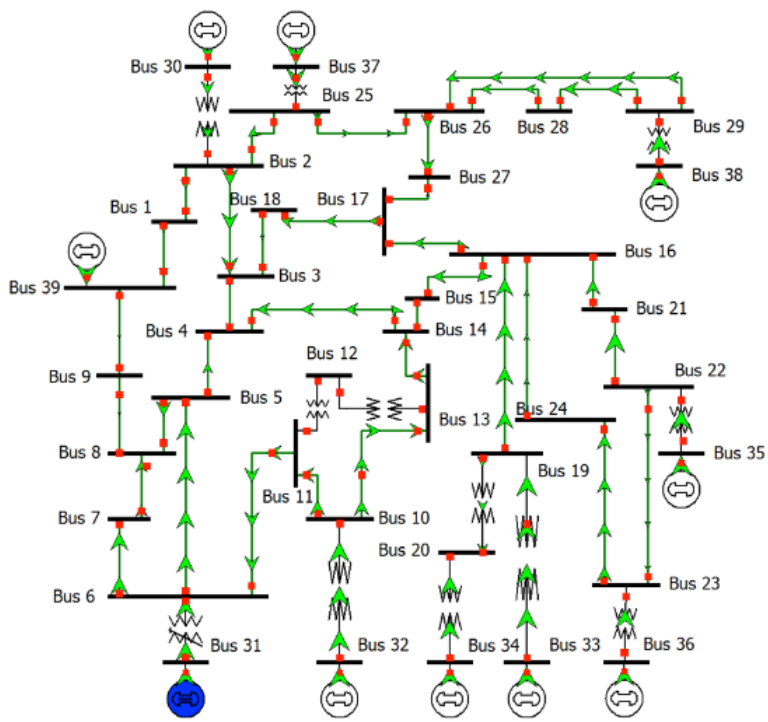


Figure 4: Illustration of the IEEE 39-bus system [28] obtained from `https://icseg.iti.illinois.edu/ieee-39-bus-system/`..

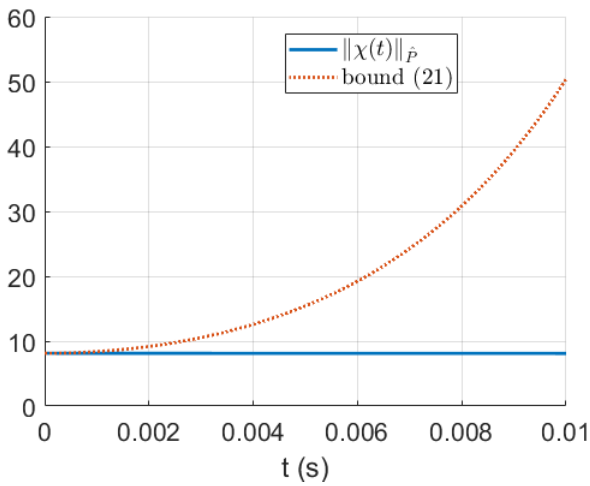
This system is comprised of 29 load buses (1 to 29 on Figure 4) and 10 generator buses (30 to 39 on Figure 4). The state of the load buses is described solely by their phase angles $\{\delta_i\}_{i \in [\![1,29]\!]}$, while the state of the generators is composed of phase angles $\delta_{i \in [\![30,39]\!]}$ and frequencies $\omega_{i \in [\![30,39]\!]}$, which leads to 49 states. Only the generator buses possess a control input u_g. Following [29], we study the linearized equations after adjustment for the reference bus, chosen to be the first generator, i.e., bus 30. The state vector is then

$$x = \big(\{\delta_i - \delta_{30}\}_{i \in [\![1,29]\!] \cup [\![31,39]\!]}, \{\omega_i\}_{i \in [\![30,39]\!]}\big) \in \mathbb{R}^{48}.$$

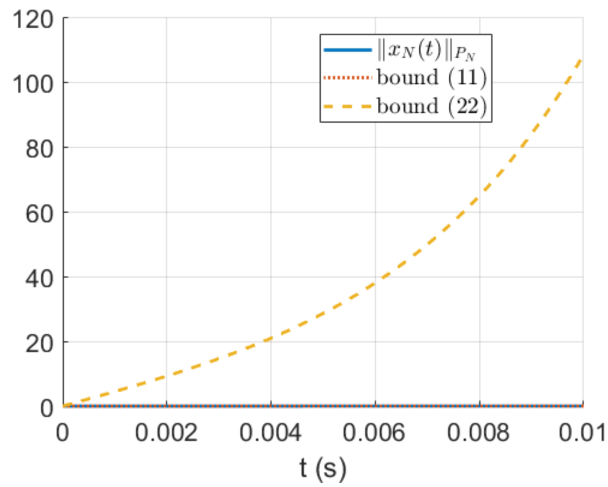
After a cyber-attack, the network controller loses control authority over the actuator of generator bus 39, i.e., $x_N = \big(\delta_{39}\ \omega_{39}\big)^\top$ and $w_N = u_{39}$. Following [29], the malfunctioning dynamics are then

$$\begin{pmatrix} \dot{\delta}_{39}(t) \\ \dot{\omega}_{39}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -18.63 & -11.22 \end{pmatrix} \begin{pmatrix} \delta_{39}(t) \\ \omega_{39}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0.222 \end{pmatrix} w_N(t) + D_{N,-}\chi(t). \tag{27}$$

As in Section 5.2, we choose the initial states to be $\chi(0) = \mathbf{1}_{46}$ and $x_N(0) = (00)^\top$. Since A_N is Hurwitz and $B_N = 0$, the assumptions of Proposition 7 are satisfied. Additionally, pair $(\hat{A} + \hat{D}, \hat{B})$ is controllable so we can find a stabilizing gain matrix K for the network dynamics. However, we cannot apply Proposition 10 because the stability condition $\gamma\gamma_N < \alpha\alpha_N$ is not satisfied. Indeed, $\gamma\gamma_N = 6.3 \times 10^4$, while $\alpha\alpha_N = 5.7 \times 10^{-3}$. This magnitude difference leads to the exponential divergence of bound (21) and of bound (22), as seen on Figure 5(a) and 5(b), respectively.



(a) Simulation of network state χ of the IEEE 39-bus system with exponentially diverging bound (21).



(b) Simulation of malfunctioning state x_N of the IEEE 39-bus system with tight bound (11) and exponentially diverging bound (22).

Figure 5: Illustration of bounds (11), (21) and (22) on the states χ and x_N.

Note that bound (11) is much tighter than (22) on Figure 5(b) because it is not built from the exponentially diverging bound (21). In fact, bound (11) remains a reasonable bound for malfunctioning state x_N over a much longer time horizon as illustrated on Figure 6.

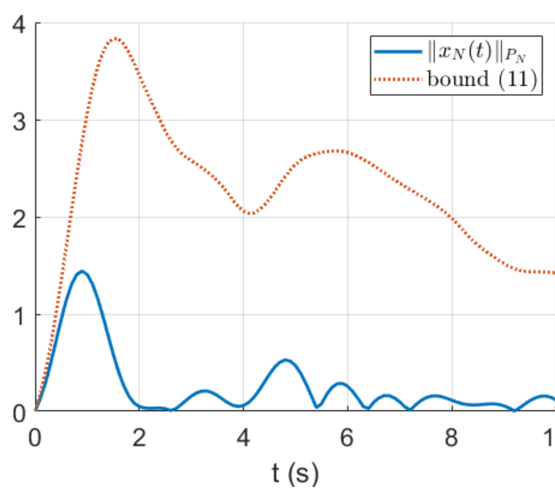


Figure 6: Simulation of malfunctioning state x_N of the IEEE 39-bus system with tight bound (11).

As before, sufficient condition $\sup_{t \geq 0} b(t) \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$ of Proposition 10 cannot tell whether linear feedback $\hat{u}$ is admissible. However, the choice of K ensures admissibility $\max_{i,t}|K\chi_i(t)| \leq 1$ as shown on Figure 7.

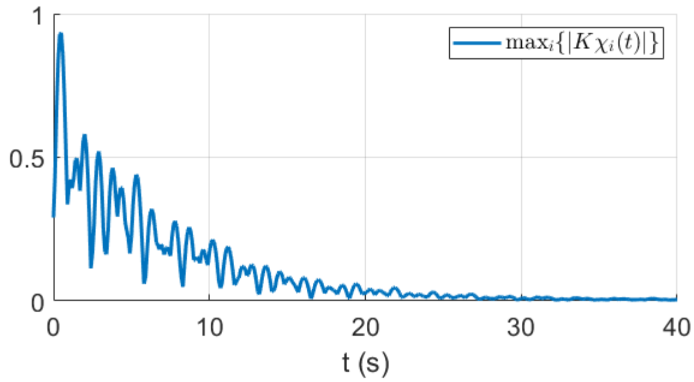


Figure 7: Maximal component of the linear feedback $\hat{u}(t) = -K\chi(t)$.

Let us delve a bit deeper into the exponential divergence of bound (21). As mentioned previously, bound (21) is not tight because $\gamma\gamma_N = 6.3 \times 10^4$ is orders of magnitude larger than $\alpha\alpha_N = 5.7 \times 10^{-3}$, whereas the stability condition of Proposition 10 calls for $\gamma\gamma_N < \alpha\alpha_N$. As discussed after Theorem 4, this condition carries the intuition that the perturbations arising from the coupling between x_N and χ should be weaker than their respective stability. Despite having $\gamma\gamma_N >> \alpha\alpha_N$, the coupling does not destabilize states x_N and χ, which are both bounded, as shown on Figure 8.

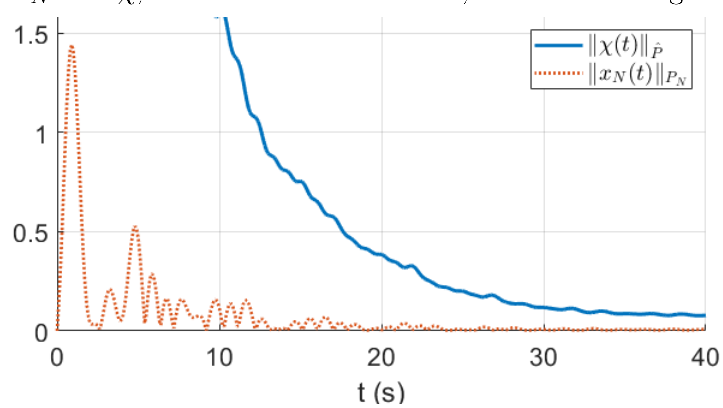


Figure 8: Simulation of network state χ and malfunctioning state x_N of the IEEE 39-bus system.

Since the coupling does not destabilize states χ and x_N, the violation of stability condition $\gamma\gamma_N < \alpha\alpha_N$ is in fact due to the failure of parameters γ and γ_N to characterize the coupling between states χ and x_N. As shown on Figure 4 each bus is only connected to a small number of other buses. Then, matrix $\hat{D}$ is almost entirely composed of zeros except for a handful of terms per row. Because of this strong coupling with very few nodes, constants γ and γ_N are very large. However, the sparsity of matrix $\hat{D}$ results in weak coupling of states χ and x_N, rendering γ and γ_N overly conservative. We have the intuition that choosing a different norm reflecting the sparsity of matrix $\hat{D}$ would lower the values of γ and γ_N. Doing so would significantly and non-trivially alter all the proofs of Section 4.

6 Conclusion

This paper investigated the resilient stabilizability of linear networks enduring a loss of control. We first saw that the overall stabilizability of networks composed exclusively of resilient subsystems depends only on their interconnection. Then, we focused on networks losing control authority over a nonresilient subsystem. In this scenario, we showed that under some conditions, the state of underactuated networks can remain bounded and the state of fully actuated networks can be stabilized. We were able to quantify the maximal magnitude of undesirable inputs that can be applied to a nonresilient subsystem without destabilizing the rest of the network.

We are considering several avenues of future work. First, building on the nonlinear resilience theory of [18], we would like to extend our approach to nonlinear networks. Doing so would allow us to study the true nonlinear dynamics of power systems, including the IEEE 39-bus system. Second, following the discussion at the end of Section 5.3 we want to extend this theory to different matrix norms to provide tighter bounds for sparse coupling matrices. The last avenue of future work would be to relax the assumption of real-time knowledge of the undesirable inputs by the controller. Doing so would allow to account for actuation delays and can possibly be accomplished following the techniques introduced in [18].

A Supporting lemmata

Lemma 1. *$\mu_{\bar{B}}(A) \geq \mu(A, \bar{B})$.*

Proof. Define sets

$$\mathcal{S}_A := \big\{(\Delta A, 0) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} : (A + \Delta A, \bar{B} + 0) \text{ is uncontrollable}\big\},$$
$$\mathcal{S}_{AB} := \big\{(\Delta A, \Delta\bar{B}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} : (A + \Delta A, \bar{B} + \Delta\bar{B}) \text{ is uncontrollable}\big\}.$$

Notice that $\mathcal{S}_A \subseteq \mathcal{S}_{AB}$. Therefore,

$$\min\big\{\|\Delta A, \Delta\bar{B}\| : (\Delta A, \Delta\bar{B}) \in \mathcal{S}_A\big\} \geq \min\big\{\|\Delta A, \Delta\bar{B}\| : (\Delta A, \Delta\bar{B}) \in \mathcal{S}_{AB}\big\},$$

i.e., $\mu_{\bar{B}}(A) \geq \mu(A, \bar{B})$. □

The following result relates the norms induced by two positive definite matrices of different sizes.

Lemma 2. *Let $D \in \mathbb{R}^{m\times n}$, $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$. If $P = P^\top \succ 0$ and $Q \succ 0$, then $\|Dx\|_P \leq \|x\|_Q \sqrt{\frac{\lambda_{max}^{D^\top PD}}{\lambda_{min}^{Q}}}$ for all $x \in \mathbb{R}^n$.*

Proof. Applying the Rayleigh quotient inequality [23] to symmetric matrices Q and $D^\top PD$ yields,

$$\lambda_{min}^{Q} \leq \frac{x^\top Qx}{x^\top x} \quad \text{and} \quad \frac{x^\top D^\top PDx}{x^\top x} \leq \lambda_{max}^{D^\top PD},$$

for all $x \in \mathbb{R}^n$, $x \neq 0$. Then,

$$\begin{aligned}\|Dx\|_P = \sqrt{x^\top D^\top PDx} &\leq \sqrt{\lambda_{max}^{D^\top PD}}\sqrt{x^\top x} \\ &\leq \sqrt{\frac{\lambda_{max}^{D^\top PD}}{\lambda_{min}^{Q}}}\sqrt{x^\top Qx} = \|x\|_Q \sqrt{\frac{\lambda_{max}^{D^\top PD}}{\lambda_{min}^{Q}}}.\end{aligned}$$

□

Since $(x,y) \mapsto x^\top P y$ defines a scalar product for any $P \succ 0$, it verifies the Cauchy-Schwarz inequality [23]. We provide here a more constructive proof of this result for the reader.

Lemma 3 (Cauchy-Schwarz inequality for the P-norm)**.** *Let $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. Then, $x^\top P y \leq \|x\|_P \|y\|_P$.*

Proof. Since $P \succ 0$, there exists a matrix $M \in \mathbb{R}^{n\times n}$ such that $P = M^\top M$ [23]. Then,

$$x^\top P y = x^\top M^\top M y = (Mx)^\top M y \leq \|Mx\| \, \|My\|,$$

by the Cauchy-Schwarz inequality applied to the Euclidean norm on $\mathbb{R}^n$ [23]. Note that

$$\|Mx\| = \sqrt{(Mx)^\top M x} = \sqrt{x^\top M^\top M x} = \sqrt{x^\top P x} = \|x\|_P.$$

Similarly, $\|My\| = \|y\|_P$. Thus, $x^\top P y \leq \|x\|_P \|y\|_P$. □

We now show how the non-resilience of subsytem (4) translates to a positive $z_{max}^{P_N}$.

Lemma 4. *With $P_N \succ 0$ and $z_{max}^{P_N} = \max\limits_{w_N \in \mathcal{W}_N} \big\{ \min\limits_{u_N \in \mathcal{U}_N} \|C_N w_N + B_N u_N\|_{P_N} \big\}$, we have $-C_N \mathcal{W}_N \nsubseteq B_N \mathcal{U}_N \iff z_{max}^{P_N} > 0$.*

Proof. If $-C_N \mathcal{W}_N \subseteq B_N \mathcal{U}_N$, then for all $w_N \in \mathcal{W}_N$, there exists $u_N \in \mathcal{U}_N$ such that $C_N w_N + B_N u_N = 0$. Hence, $\min\limits_{u_N \in \mathcal{U}_N} \big\{ \|C_N w_N + B_N u_N\|_{P_N} \big\} = 0$ for all $w_N \in \mathcal{W}_N$, i.e., $z_{max}^{P_N} = 0$.

On the other hand, if $-C_N \mathcal{W}_N \nsubseteq B_N \mathcal{U}_N$, there exists $w_N \in \mathcal{W}_N$ such that $C_N w_N + B_N u_N \neq 0$ for all $u_N \in \mathcal{U}_N$. The function $u_N \mapsto \|C_N w_N + B_N u_N\|_{P_N}$ is continuous, nonnegative and $\mathcal{U}_N$ is compact, hence it reaches a minimum which cannot be null on $\mathcal{U}_N$, i.e., $\min\limits_{u_N \in \mathcal{U}_N} \big\{ \|C_N w_N + B_N u_N\|_{P_N} \big\} > 0$. Then, $z_{max}^{P_N} > 0$. □

Lemma 5. *Detailed calculations for the proof of Proposition 8.*

Proof. We first study the linear homogeneous differential equation associated with (15), i.e.,

$$\ddot{s}(t) + (\alpha - \alpha_N)\dot{s}(t) - \gamma\gamma_N s(t) = 0. \tag{28}$$

Solutions of (28) can be written as $s_h(t) = e^{rt}$ with $r \in \mathbb{C}$. Plugging s_h in (28) leads to the quadratic equation $r^2 + (\alpha - \alpha_N)r - \gamma\gamma_N = 0$ after diving by e^{rt}. The solutions of this quadratic equation are $r_\pm = \frac{1}{2}\big(\alpha_N - \alpha \pm \sqrt{(\alpha - \alpha_N)^2 + 4\gamma\gamma_N}\big)$. Notice that the discriminant is nonnegative, since $\gamma \geq 0$ and $\gamma_N \geq 0$, so both $r_\pm \in \mathbb{R}$. We also need a particular solution of the non-homogeneous equation (15). We take $p \in \mathbb{R}$ such that $s_p(t) = p e^{\alpha_N t}$ and plug it in (15) to obtain

$$\big(p\alpha_N^2 + (\alpha - \alpha_N)p\alpha_N - \gamma\gamma_N p - \gamma z_{max}^{P_N} + \alpha_N b_{min}^{\hat{P}}\big)e^{\alpha_N t} = 0,$$

$$\text{i.e.,} \quad p = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha_N^2 + (\alpha - \alpha_N)\alpha_N - \gamma\gamma_N} = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha\alpha_N - \gamma\gamma_N}.$$

Let us first treat the case where $\alpha\alpha_N \neq \gamma\gamma_N$, so that p is well-defined. In this case, the general solution of (15) is $s(t) = p e^{\alpha_N t} + h_+ e^{r_+ t} + h_- e^{r_- t}$ with $h_\pm \in \mathbb{R}$ two constants to choose. Since we obtained our solution by solving $\ddot{s}(t) = \frac{df}{dt}\big(t, s(t)\big)$ instead of $\dot{s}(t) = f\big(t, s(t)\big)$, we have an additional initial condition to satisfy: $\dot{s}(0) = f\big(0, s(0)\big)$.

Now we can apply the Comparison Lemma of [27] stating that if $\dot{s}(t) = f\big(t, s(t)\big)$, f is continuous in t and locally Lipschitz in s and $s(0) = v(0)$, then $\dot{v}(t) \leq f\big(t, v(t)\big)$ implies $v(t) \leq s(t)$ for all $t \geq 0$. Using $\|\chi(t)\|_{\hat{P}} = y(t) = e^{-\alpha_N t} v(t) \leq e^{-\alpha_N t} s(t)$, we finally obtain (12). To determine the value of the constants $h_\pm$, we use the initial conditions $s(0) = v(0) = y(0)$ and $\dot{s}(0) = f\big(0, s(0)\big)$, i.e.,

$$p + h_+ + h_- = \|\chi(0)\|_{\hat{P}} \quad \text{and} \quad \alpha_N p + h_+ r_+ + h_- r_- = (\alpha_N - \alpha)\|\chi(0)\|_{\hat{P}} - b_{min}^{\hat{P}} + \gamma \|x_N(0)\|_{P_N}.$$

We can solve these equations as

$$h_\pm = \frac{(\alpha_N - \alpha - r_\mp)\|\chi(0)\|_{\hat{P}} + \gamma\|x_N(0)\|_{P_N} - b_{min}^{\hat{P}} + (r_\mp - \alpha_N)p}{\pm\sqrt{(\alpha - \alpha_N)^2 + 4\gamma\gamma_N}}.$$

In the case $\alpha\alpha_N = \gamma\gamma_N$, the discriminant of the quadratic equation arising from the homogeneous differential equation is $(\alpha - \alpha_N)^2 + 4\alpha\alpha_N = (\alpha + \alpha_N)^2$, which yields $r_+ = \alpha_N$ and $r_- = -\alpha$. Hence $e^{\alpha_N t}$ is an homogeneous solution and cannot be a particular solution of the non-homogeneous differential equation (15). Instead, we try $s_p(t) = pte^{\alpha_N t}$ as a particular solution. We calculate its derivatives $\dot{s}_p(t) = p(1+\alpha_N t)e^{\alpha_N t}$, $\ddot{s}_p(t) = p(2\alpha_N + \alpha_N^2 t)e^{\alpha_N t}$ and plug it in (15). After dividing by $e^{\alpha_N t}$ we obtain

$$\begin{aligned} 0 &= p(2\alpha_N + \alpha_N^2 t) + (\alpha - \alpha_N)p(1+\alpha_N t) - \alpha\alpha_N pt - \gamma z_{max}^{P_N} + \alpha_N b_{min}^{\hat{P}} \\ &= p(2\alpha_N + \alpha - \alpha_N) + pt\big(\alpha_N^2 + \alpha_N(\alpha - \alpha_N) - \alpha\alpha_N\big) - \gamma z_{max}^{P_N} + \alpha_N b_{min}^{\hat{P}}, \end{aligned}$$

i.e., $p = \frac{\gamma z_{max}^{P_N} - \alpha_N b_{min}^{\hat{P}}}{\alpha + \alpha_N}$. In this case p is well-defined since $\alpha > 0$ and $\alpha_N > 0$. The general solution is then $s(t) = pte^{\alpha_N t} + h_+ e^{\alpha_N t} + h_- e^{-\alpha t}$ with $h_\pm \in \mathbb{R}$ two constants. Applying the Comparison Lemma of [27] as above, we obtain $\|\chi(t)\|_{\hat{P}} = y(t) = e^{-\alpha_N t}v(t) \leq e^{-\alpha_N t}s(t)$, which yields (13). The initial conditions $s(0) = y(0)$ and $\dot{s}(0) = f\big(0, s(0)\big)$ lead to

$$h_+ + h_- = \|\chi(0)\|_{\hat{P}} \quad \text{and} \quad p + h_+\alpha_N - h_-\alpha = (\alpha_N - \alpha)\|\chi(0)\|_{\hat{P}} - b_{min}^{\hat{P}} + \gamma\|x_N(0)\|_{P_N}.$$

We can solve these equations as

$$h_\pm = \frac{\frac{1}{2}\big(-\alpha_N - \alpha \pm 3(\alpha - \alpha_N)\big)\|\chi(0)\|_{\hat{P}} \mp \gamma\|x_N(0)\|_{P_N} \pm b_{min}^{\hat{P}} \pm p}{\alpha_N + \alpha}.$$

□

We now describe how to compute the supremum of $b(t)$ which is required to employ Proposition 10. Following Proposition 10, we only investigate the case $\gamma\gamma_N < \alpha\alpha_N$ or equivalently $r_+ < \alpha_N$.

Lemma 6. *If $r_+ < \alpha_N$, then Algorithm 1 provides $\sup\{b(t) : t \geq 0\}$ defined in (20).*

Proof. Recall that $b(t) := p + h_+ e^{(r_+ - \alpha_N)t} + h_- e^{(r_- - \alpha_N)t}$ for all $t \geq 0$. Since $r_\pm - \alpha_N < 0$, both exponential terms converge to 0. Thus, $b(t)$ is bounded on $t \geq 0$ and has at most one extremum, except in the trivial case $h_+ = h_- = 0$ where $b(t) = p$ for all $t \geq 0$.

Otherwise, to obtain the supremum of $b(t)$ we first calculate its derivative $\dot{b}(t) = h_+(r_+ - \alpha_N)e^{(r_+ - \alpha_N)t} + h_-(r_- - \alpha_N)e^{(r_- - \alpha_N)t}$. Then, solving $\dot{b}(T) = 0$ yields the location of the single extrema of $b(t)$ at $T = \frac{1}{r_- - r_+}\ln\left(\frac{h_+(r_+ - \alpha_N)}{h_-(\alpha_N - r_-)}\right)$. If $\ddot{b}(T) > 0$, then the extrema at T is a minimum. Since $b(T)$ is the only extremum of b, the maximum of b is then reached at 0 or at infinity. Note that $b(0) = p + h_+ + h_-$ and $\lim\limits_{t\to\infty} b(t) = p$. Otherwise, $\ddot{b}(T) < 0$ leads to the conclusion that $b(T)$ is the maximum of b. □

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