Quantitative Resilience of Linear Systems
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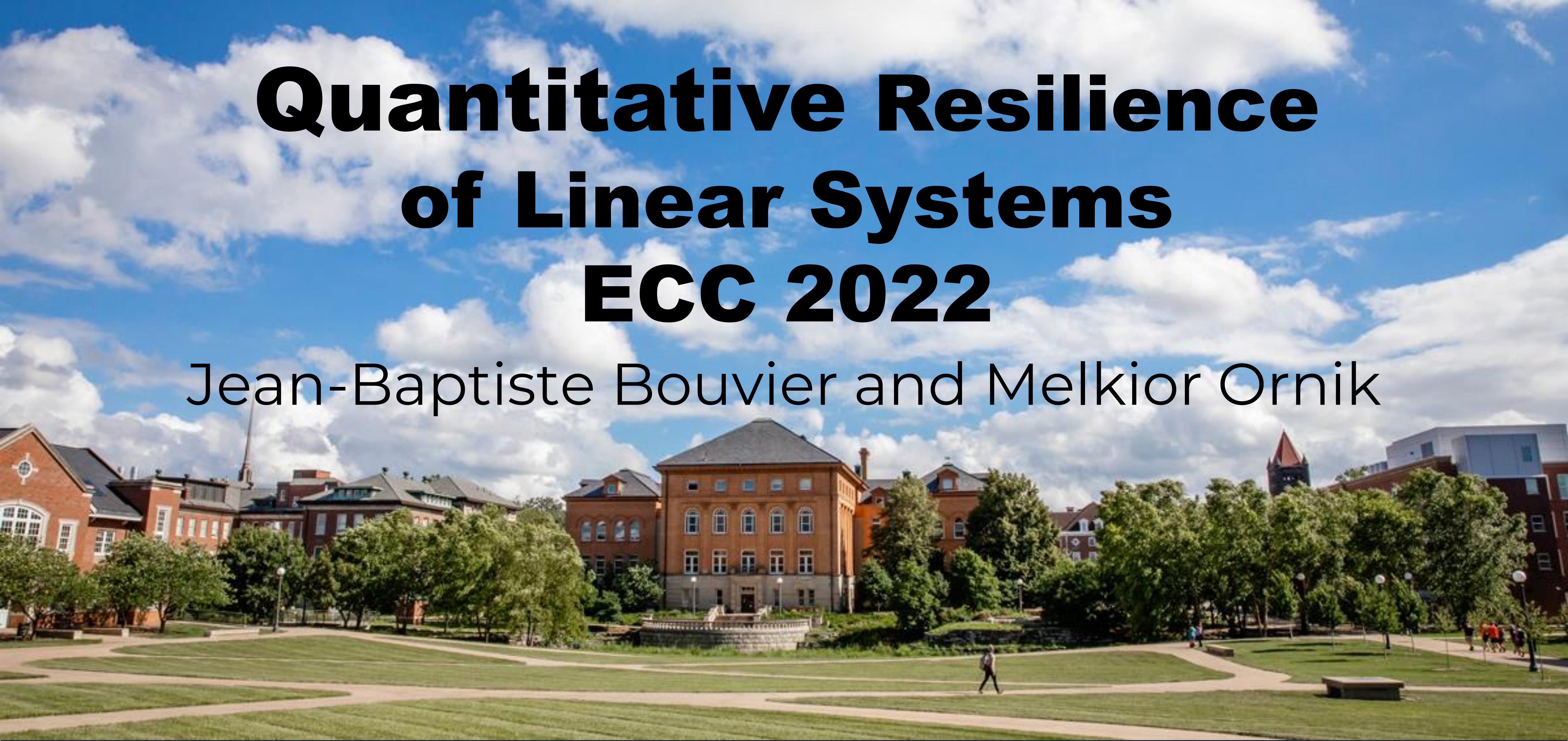

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Motivation

Nauka module (left) docked to the ISS. (Image credit: Thomas Pesquet/ESA/NASA)

Loss of control authority over an actuator, that now produces uncontrolled and possibly **undesirable inputs**.

Sensors measuring in real-time all actuators inputs.

Can we still stabilize the system?

Preliminaries

$\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t), \quad x(0) = x_0, \quad \bar{u}(t) \in \bar{U}.$

Loss of control authority over some actuators of $\bar{B}$:

separate controls u from undesirable inputs w as $\bar{u} = (u, w)$ and $\bar{B} = [B\ C]$.

$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \quad x(0) = x_0, \quad u(t) \in U, \quad w(t) \in W.$

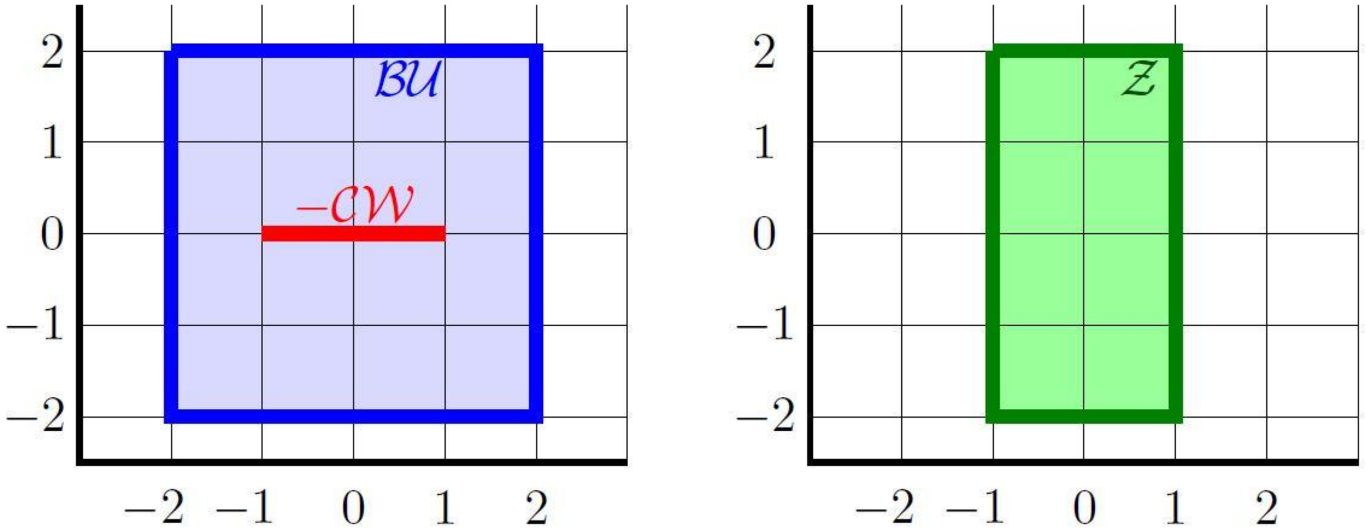
The system is **resiliently stabilizable** if for all $x_0 \in \mathbb{R}^n$ and all $w(\cdot) \in W$ there exists $u(\cdot) \in U$ and $T > 0$ such that $x(T) = 0$.

Resilience of Linear Systems

Let $\mathcal{Z} = B\mathcal{U} \ominus (-C\mathcal{W})$ = $\{z \in B\mathcal{U}\colon\ z - Cw \in B\mathcal{U}\ \text{ for all }\ w \in \mathcal{W}\}$ the Minkowski difference of $B\mathcal{U} = \{Bu : u \in \mathcal{U}\}$ and $-C\mathcal{W} = \{-Cw : w \in \mathcal{W}\}$.

$\mathcal{Z}$ is the set of controls remaining after counteracting any undesirable input Cw.

$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,
$\mathcal{U} = [-1, 1]^4$ and $\mathcal{W} = [-1, 1]$:



Resilience of Linear Systems

Hájek's Duality theorem: $\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$ is resiliently stabilizable if and only if $\dot{x}(t) = Ax(t) + z(t)$ is stabilizable, with $z(t) \in \mathcal{Z}$.

Resilient stabilizability is transformed into **bounded stabilizability.**

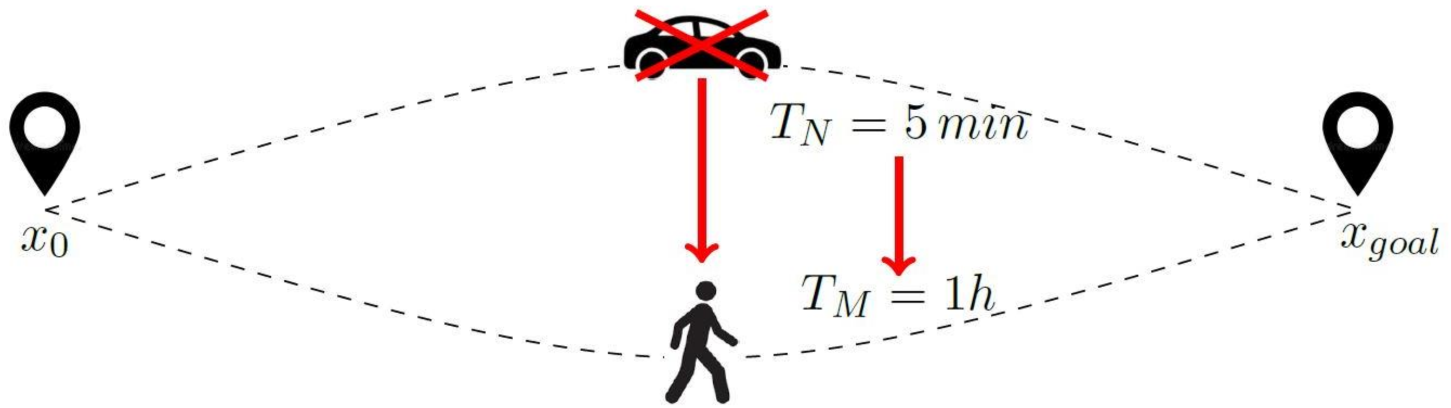
Using Brammer's controllability theory we obtain a sufficient condition for resilient stabilizability:

If $Re\big(\lambda(A)\big) \leq 0$ and $0 \in int(\mathcal{Z})$, the system is resiliently stabilizable.

O. Hájek, Duality for differential games and optimal control, Mathematical Systems Theory, 1974.
R. Brammer, Controllability in linear autonomous systems with positive controllers, SIAM Journal on Control, 1972.

Quantitative Resilience

Resilience only guarantees reachability despite a loss of control authority. It does not quantify the impact of this malfunction.



$r_q = \frac{T_N}{T_M} \approx 0.08$

How to evaluate the **quantitative resilience $\boldsymbol{r_q}$** of the system?

Quantitative Resilience

Nominal reach time $T_N^*(x_0) = \inf\limits_{\bar{u}\,\in\,\bar{U}} \{\, T \colon x(T) = 0\}$ for $\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t)$ and $\bar{U} = [-1, 1]^{m+p}$.

Malfunctioning reach time $\mathrm{T}_{\mathrm{M}}^*(x_0) = \sup\limits_{w\,\in\,W} \{\, \inf\limits_{u\,\in\,U} \{\, T \colon x(T) = 0\}\}$

for $\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$, $U = [-1, 1]^m$ and $W = [-1, 1]^p$.

Ratio of reach times $t(x_0) = \dfrac{T_M^*(x_0)}{T_N^*(x_0)}$ for $x_0 \in \mathbb{R}^n$.

Quantitative resilience $r_q = \inf\limits_{x_0 \in\, \mathbb{R}^n} \dfrac{T_N^*(x_0)}{T_M^*(x_0)} = \dfrac{1}{\sup\limits_{x_0\,\in\,\mathbb{R}^n} t(x_0)} \leq 1.$

Framework

For the nominal reach time $T_N^*(x_0)$, the time-optimal control input $\bar{u}^*$ is bang-bang, but T_N^* has no closed-form expression.

For the malfunctioning reach time $T_{\mathrm{M}}^*(x_0)$, the framework is even more complex due to the interactions between u and w.
How to find $u^*(t, w(t))$? For what $w(t)$?
Maximize $\dot{x}(t)$? Maximize $\langle \dot{x}(t), -x(t) \rangle$? Assume that w is constant?

But we need u^* and w^* to be both bang-bang to make $T_{\mathrm{M}}^*(x_0)$ time-optimal and comparable to $T_{\mathrm{N}}^*(x_0)$.

JP. LaSalle, Time optimal control systems, Proceedings of the National Academy of Sciences, 1959.
L. Neustadt, Synthesizing time optimal control systems, Journal of Mathematical Analysis and Applications, 1960.

Framework

Closed-loop capture: $u^*(t)$ knows $w^*([0,t])$ and solves Isaac's main equation: differential game equivalent of HJB equation (usually intractable PDE). Instead, they give sub-optimal solutions, their T_M^* cannot be compared with time-optimal T_N^*.

Open-loop capture: controller knows w^* in advance by assuming that w is the worst bang-bang input.

We follow Sakawa's pursuit-evasion game framework so that both T_N^* and T_{M}^* are time-optimal. They are both achieved bang-bang inputs but have no closed-form solutions.

Y. Sakawa, Solution of linear pursuit-evasion games, SIAM Journal on Control, 1970.
W. Borgest and P. Varaiya, Target Function Approach to Linear Pursuit Problems, IEEE Transactions on Automatic Control, 1971.

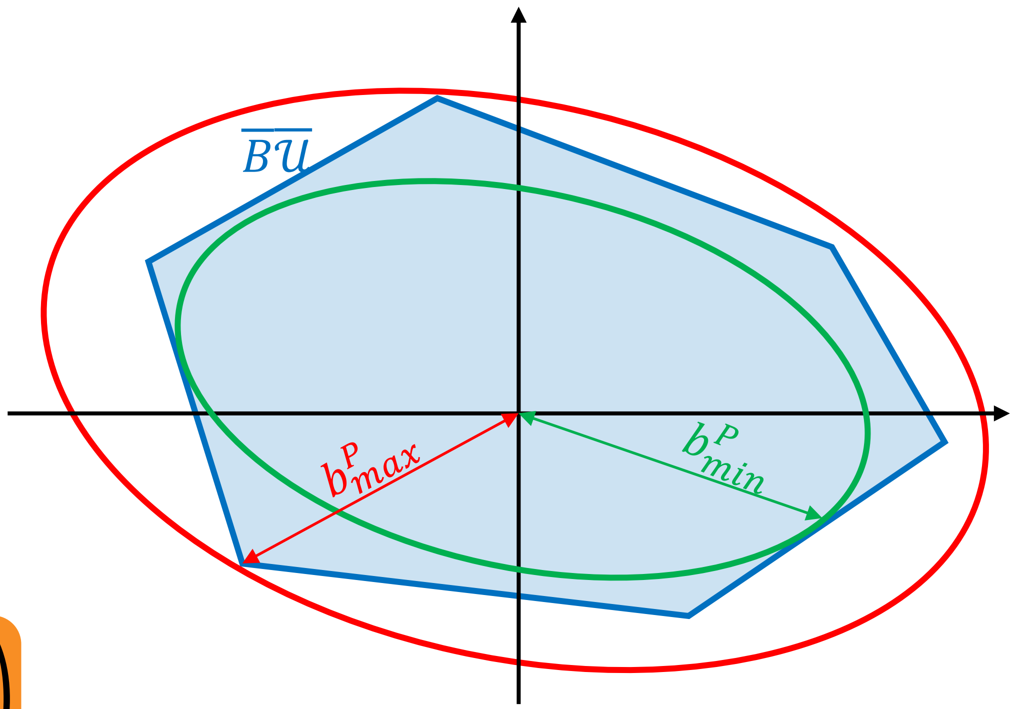
Bounding Quantitative Resilience

If $\dot{x} = Ax + \bar{B}\bar{u}$ is stabilizable and A is Hurwitz,

then $\boxed{T_N^*(x_0) \geq 2\, \frac{\lambda_{min}^P}{\lambda_{max}^Q}\, ln\left(1 + \frac{\lambda_{max}^Q ||x_0||_P}{2\lambda_{min}^P b_{max}^P}\right)}$,

$P \succ 0, \quad Q \succ 0, \quad PA + A^\top P = -Q, \quad ||x||_P^2 = x^\top P x$

and $\quad b_{max}^P = \max\left\{||\bar{B}\bar{u}||_P : \bar{u} \in \bar{\mathcal{U}}\right\}$.



If $\bar{B}$ is full-rank, $\boxed{T_N^*(x_0) \leq 2\, \frac{\lambda_{max}^P}{\lambda_{min}^Q}\, ln\left(1 + \frac{\lambda_{min}^Q ||x_0||_P}{2\lambda_{max}^P b_{min}^P}\right)}$

with $b_{min}^P = \min\left\{||\bar{B}\bar{u}||_P : \bar{u} \in \partial\bar{\mathcal{U}}\right\}$.

Bounding Quantitative Resilience

If $\dot{x} = Ax + Bu + Cw$ is resiliently stabilizable, then

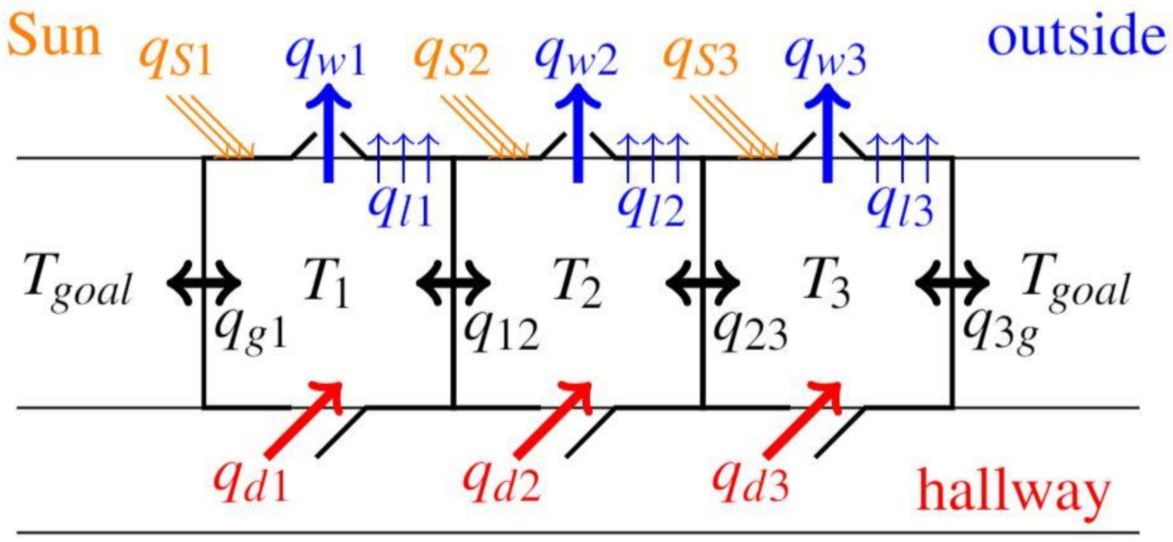
$$2\,\frac{\lambda^P_{min}}{\lambda^Q_{max}}\, ln\left(1 + \frac{\lambda^Q_{max}||x_0||_P}{2\lambda^P_{min} z^P_{max}}\right) \leq T^*_M(x_0) \leq 2\,\frac{\lambda^P_{max}}{\lambda^Q_{min}}\, ln\left(1 + \frac{\lambda^Q_{min}||x_0||_P}{2\lambda^P_{max} z^P_{min}}\right),$$

with $z^P_{min} = \min\left\{||z||_P : z \in \partial\mathcal{Z}\right\}$ and $z^P_{max} = \max\left\{||z||_P : z \in \mathcal{Z}\right\}$.

Recall that $r_q = \inf\limits_{x_0 \in \mathbb{R}^n} \frac{T^*_N(x_0)}{T^*_M(x_0)}$. If $\dot{x} = Ax + Bu + Cw$ is resiliently stabilizable, then

$$min\left(\frac{\lambda^P_{min}\lambda^Q_{min}}{\lambda^P_{max}\lambda^Q_{max}}, \frac{z^P_{min}}{b^P_{max}}\right) \leq r_q \leq min\left(\frac{\lambda^P_{max}\lambda^Q_{max}}{\lambda^P_{min}\lambda^Q_{min}}, \frac{z^P_{max}}{b^P_{min}}\right).$$

Temperature Control System



Actuators:

- Central heating/AC $\quad q_h - q_{AC} = Q_{hAC}u_{hAC}$
- Door/window $\quad q_d - q_w = Q_{dw}u_{dw}$
- Sunshades/heat loss $\quad q_S - q_l = Q_{Sl}u_{Sl}$

Objective: $\mathrm{T}_1 = \mathrm{T}_2 = \mathrm{T}_3 = \mathrm{T}_{\mathrm{goal}}$.

$$mC_p\dot{T}_1 = q_{g1} + q_{12} + Q_{hAC}u_{hAC} + Q_{dw}u_{dw}^1 + Q_{Sl}u_{Sl}^1$$
$$mC_p\dot{T}_2 = -q_{12} + q_{23} + Q_{hAC}u_{hAC} + Q_{dw}u_{dw}^2 + Q_{Sl}u_{Sl}^2$$
$$mC_p\dot{T}_3 = -q_{23} + q_{g3} + Q_{hAC}u_{hAC} + Q_{dw}u_{dw}^3 + Q_{Sl}u_{Sl}^3$$

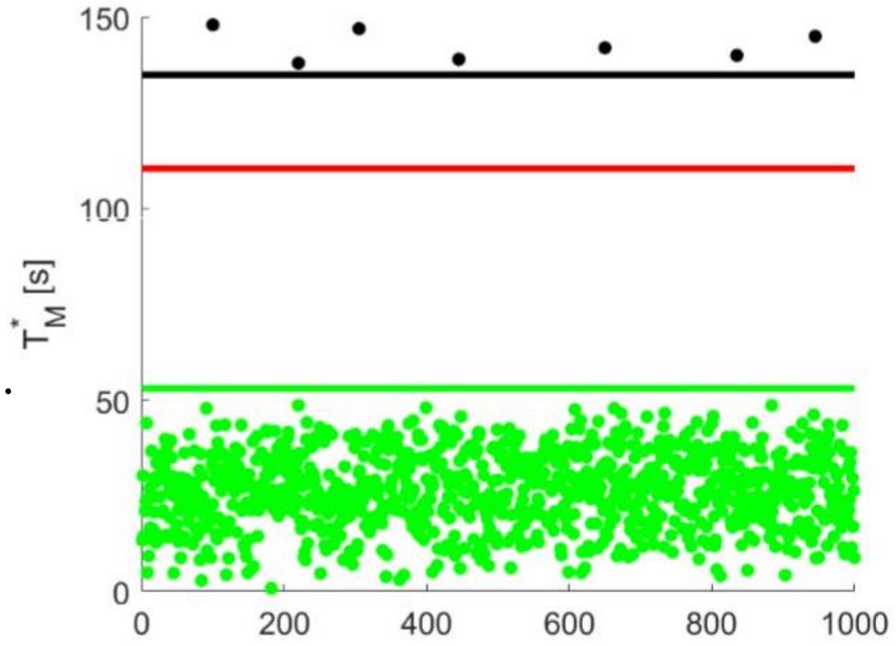
$$x(t) = \begin{pmatrix} T_1(t) - T_{goal} \\ T_2(t) - T_{goal} \\ T_3(t) - T_{goal} \end{pmatrix} \qquad x_0 = \begin{pmatrix} 0.8°\mathrm{C} \\ 0.7°\mathrm{C} \\ 0.9°\mathrm{C} \end{pmatrix}$$

$$\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t)$$

Temperature Control System

Loss of control over door 2.
Numerical computation of $T_N^*(x_0)$ and $T_M^*(x_0)$: red line
Analytical bounds for 1000 random pairs (P, Q) : green and black dots
and for the ellipsoid approximations of $\bar{B}\bar{\mathcal{U}}$ and $\mathcal{Z}$: green and black lines

$35s \leq T_N^*(x_0) = 42s \leq 54s \qquad 53s \leq T_M^*(x_0) = 110s \leq 135s.$



$$t(x_0) = \frac{T_M^*(x_0)}{T_N^*(x_0)} = 2.6 \leq 3.8$$

Rooms can take up to 2.6 times longer to reach T_{goal} from x_0.

$$0.097 \leq r_q \leq 2.79$$

From any x_0 they can take up to $\frac{1}{0.097} = 10.3$ times longer.

Conclusion and Future Work

We established:

- sufficient conditions for resilient stabilizability of linear systems with bounded inputs;
- bounds on the nominal and malfunctioning reach times to quantify resilience of linear systems.

We will work on

- removing the full-rank requirement of B from resilience conditions;
- necessary and sufficient conditions for resilience;
- resilience of networks to partial loss of control authority.

Thank you for your attention